Stability of Radner Equilibria with Respect to Small Frictions*

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Abstract

We study risk-sharing equilibria with trading subject to small proportional transaction costs. We show that the frictionless equilibrium prices also form an "asymptotic equilibrium" in the small-cost limit. To wit, there exist asymptotically optimal policies for all agents and a split of the trading cost according to their risk aversions for which the frictionless equilibrium prices still clear the market. Starting from a frictionless equilibrium, this allows to study the interplay of volatility, liquidity, and trading volume.

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1 Introduction

Frictions such as transaction costs play a crucial role in many parts of financial theory, cf., e.g., [55] for an overview. A prime example is the discussion about financial transaction taxes. The study of such regulatory measures naturally requires a general equilibrium setting, in order to assess how changes in the playing field affect the interplay of liquidity, volatility, and trading volume.

The analysis of such models is challenging, however, because frictions exacerbate the tractability issues inherent in general equilibrium settings. Indeed, frictions drastically complicate individual decision making. Moreover, representative agents can only be introduced to simplify the analysis in precisely those settings where no-trade equilibria obtain and frictions are irrelevant.

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Accordingly, most of the extant literature has focused either on numerical methods,[1] or on models with various simplifying assumptions.[2] This is in stark contrast to recent progress in the analysis of partial equilibrium models with *small* frictions, where explicit asymptotically optimal policies are now available in very general frameworks.[3]

The present study brings these asymptotic methods to bear on general equilibrium models. This is inspired by the work of Lo, Mamaysky and Wang [48] on equilibria with small fixed trading costs.[4] Like in their model, we also focus on a risk-sharing equilibrium, where two heterogeneous agents with constant absolute risk aversion receive random endowments and trade shares of a correlated, dividend-paying asset to hedge their exposures. However, unlike [48], we consider rather general dynamics that allow for unspanned aggregate risks as well as for random fluctuations of both asset prices and agents' positions in equilibrium. (In contrast, the equilibrium price in [48] is constant. Other equilibrium models such as [68] or [14, 66, 67] lead to equilibria with no or smooth trading, for which the effect of small frictions is negligible exactly or at the leading order, respectively.) Moreover, we also determine the equilibrium interest rate endogenously, by clearing the market for the consumption good. This is made possible by explicitly modeling "where the transaction costs go" in equilibrium – by also solving the optimal consumption problem of the receiver of the transaction payments.[5] This in turn allows to match absolutely continuous inflows due to endowments and dividends with the outflows due to the consumption; this would not be possible if the singular transaction payments disappear from the model. This issue does not arise if either the bank account does not have to clear as in [48], trading strategies are smooth as in [66, 67, 69], or time is discrete as in [8, 9]. An interesting – but hard – question is whether market clearing could also be achieved via singular dynamics of the interest rate as in the frictionless model of [43]. However, in any case, modeling "where the transaction costs go" gives the model a stronger general equilibrium flavor and opens the door to the welfare analysis of redistributive effects, for example.

The starting point for our analysis is a frictionless baseline equilibrium, to which we then add a small proportional transaction cost. (Other costs can be treated along the same lines, cf. Section 4.3.) We show that there exist asymptotically optimal strategies for all agents[6] and an endogenously determined split of the transaction costs according to their risk aversions,[7] such that the frictionless equilibrium prices still clear the market. In this sense, the frictionless equilibrium is stable with respect to the introduction of the marginal trading cost.

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[1] Compare, e.g., [32, 8, 9] for the recursive solution of discrete-time models.

[2] For example, [66, 67] focus on overlapping generations models with exogenous interest rates or no risky asset, respectively. [48] also do not endogenize the interest rate, and their equilibrium price is constant over time. [17] studies a one-period model where transaction costs are refunded, but agents do not internalize this. [69] focuses on a pure consumption-savings problem without risky asset.

[3] Early asymptotic analyses of the classical models of [49, 18, 24] can be found in [21, 64, 70, 35]. For more recent results in general settings, cf., e.g. [65, 56, 50, 61, 41, 42, 10, 11].

[4] The numerical results of [48, 30] also confirm the accuracy of the small-cost asymptotics for reasonable parameter values. Thereby, these expansions allow to "reveal the salient features of the problem while remaining a good approximation to the full but more complicated model" [70].

[5] One possible interpretation is a government collecting taxes; another is the operator of an exchange receiving fees.

[6] This use of approximate optimality is similar to the notion of ε-equilibria in game theory, cf. [59] and many more recent studies.

[7] If agents have identical risk aversions as in [48], each of them pays the same cost. In general, more risk-averse agents have a stronger motive to trade and are therefore willing to bear a larger share. This assumption is made for tractability; it is relaxed in [7] in a simpler model with quadratic trading costs and preferences, where only expected returns but not volatilities and interest rates are determined in equilibrium.

This robustness result is not just another envelope theorem, as retaining the frictionless benchmark policy leads to infinite transaction costs. Instead, our result states that – at the leading order – the effects of the friction are the same in general equilibrium as in a partial equilibrium, where the frictionless equilibrium price remains fixed and the transaction costs are split appropriately. This suggests that the introduction of a small financial transaction tax need not have a first-order effect on market dynamics – even though it may substantially alter the trading strategies of market participants with frequent trading needs. Moreover, our stability result implies that "welfare results" obtained in partial equilibrium models remain valid at the leading order in the present setting. An example is the finding of [42] that – in *any* model with small proportional transaction costs – 1/3 of the performance losses are a true deadweight loss, while the remaining 2/3 are just redistributed to the receiver of the transaction cost payments.

The validity of this result rests on the following assumptions. First, all agents have constant absolute risk aversions, deterministic impatience rates, and the frictionless interest rates are deterministic, so that the discounted total risk tolerances are also deterministic. This assumption crucially simplifies the analysis by abstracting from the wealth effects of transaction costs on portfolio choice. The same assumption is made in virtually all continuous-time incomplete equilibrium models, compare, e.g., [66, 67, 48, 14, 71, 13, 12, 45, 69]. It could be replaced by formally assuming that transaction costs are refunded immediately as in [17], compare Section 4.3.

Second, frictionless equilibrium asset prices are sufficiently "nice" Itô processes with nontrivial Brownian fluctuations.[8] Third, agents' optimal trading strategies are also sufficiently "nice" Itô processes with a nontrivial Brownian component. If optimal trading strategies are smooth as in [66, 67, 14], a different asymptotic regime with much smaller welfare effects of small transaction costs applies. We therefore focus on agents with "high-frequency trading needs" as in [48], who would be most affected by a financial transaction tax, for example.

The tractable models hitherto proposed in the literature do not simultaneously meet these requirements. Indeed, the equilibrium asset prices of [48] are constant, whereas the frictionless optimal trading strategies of [66, 67, 14] are smooth. Both asset prices and trading strategies are diffusive in the model of [13], but the corresponding interest rates are no longer deterministic. As a concrete example for our main result, we therefore construct a new frictionless economy that meets all of the assumptions for our main robustness result. In this concrete model, we find that the predictions of our "asymptotic equilibrium" are consistent with the insights gleaned from the expansion of an exact equilibrium in the model by Lo et al. [48].[9] However, our framework also allows to study the link between trading volume and market volatility, for example, which is not possible with constant asset prices as in [48].

The remainder of this article is organized as follows. Section 2 introduces the model, both in its frictionless baseline version and with proportional transaction costs. In Section 3, we summarize our first set of main results, which concern individual optimality with small transaction costs. The proof of these results, which extend previous results from the literature [39, 1, 10, 11] to the model features required for general equilibrium modeling, including intertemporal consumption,

[8] Asset prices with different degrees of activity could also be studied as in [61] using more advanced methods of stochastic calculus. However, this would not lead to qualitatively different results, compare Section 4.3.

[9] This suggests that relaxing individual optimality to an asymptotic version does not affect the equilibrium implications at the leading order. At higher orders, other variations of the model such as initial positions, liquidation conventions, dividend specifications, finite time horizons, etc. also play a non-negligible role. A rigorous proof of such a result in a general setting is an important direction for future research.

non-constant interest rates, dividends, as well as time-varying and stochastic transaction costs, is delegated to Appendix A. Section 4 contains the statement of our main results concerning the construction of an explicit asymptotic equilibrium with small transaction costs. It also includes a detailed discussion of several extensions and variations of the model. The corresponding proofs are delegated to Appendix B. Section 5 focuses on a concrete example for our main result, and compares it to the model of [48]. The corresponding proofs are collected in Appendices C and D. Finally, Appendix E contains auxiliary results from stochastic calculus used in the proofs.

2 Setting

2.1 Probabilistic Setup and Notation

Throughout, we fix a filtered probability space $(\Omega, \mathbb{F} = (\mathcal{F}_t)_{t\in[0,T]}, \mathcal{F}, P)$ with finite time horizon $T > 0$; the filtration $\mathbb{F}$ satisfies the usual conditions of right-continuity and completeness and the initial σ-field $\mathcal{F}_0$ is P-trivial.

$I_t = t$ denotes the identity process and $\|A\|$ the total variation of a finite-variation process A. For an adapted càdlàg process X, we set $X^*_t := \sup_{s\in[0,t]} |X_s|$. A process X is called *Itô process* if there exist an adapted *drift rate* μ^X and a predictable *(instantaneous) variance* $\Sigma^X \geq 0$ with $\int_0^T \left(|\mu^X_t| + \Sigma^X_t\right) \mathrm{d}t < \infty$ such that

$$\mathrm{d}X_t = \mu^X_t \,\mathrm{d}t + \sqrt{\Sigma^X_t}\,\mathrm{d}W_t, \tag{2.1}$$

for a P-Brownian motion W. If (2.1) holds we say that X is *driven* by W.[10] An Itô process X has *continuous variance*, if Σ^X has continuous paths, and *nonvanishing variance* if $\Sigma^X > 0$.

For Itô processes X and Y (with potentially different driving Brownian motions), $\Sigma^{X,Y}_t = \frac{\mathrm{d}\langle X,Y\rangle_t}{\mathrm{d}t}$ denotes the corresponding *(instantaneous) covariation.* Finally, for an Itô process X, we write $L(X)$ for the set of X-integrable predictable processes, and denote by $H \bullet X$ the stochastic integral of $H \in L(X)$ with respect to X.

2.2 Consumption Clock

We consider an economy with a single perishable consumption good. All quantities denoted in units of the latter are called *real.* There is a common *consumption clock* ν, a finite (deterministic) measure on $[0,T]$, which describes how agents value consumption over time. We assume that $\nu(\{0\}) = 0$, $\nu(\{T\}) > 0$ and that ν is absolutely continuous on $(0,T)$. The two main examples are $\nu(\mathrm{d}t) = \delta_T$ (terminal consumption only) or $\nu(\mathrm{d}t) = \mathbf{1}_{(0,T)}(t)\,\mathrm{d}t + \delta_T(\mathrm{d}t)$ (continuous consumption and terminal lump sum consumption).

2.3 Frictionless Market

There are two securities which are traded competitively. The first is a financial asset, henceforth called "savings account", in zero net supply. The second is a dividend-paying asset, hereafter called

[10]By Girsanov's theorem, (2.1) remains valid under equivalent probability measures $Q \approx P$ after replacing the P-drift rate μ^X with the Q-drift rate $\mu^{X,Q}$ and W with a Q-Brownian motion W^Q.

"stock", in unit net supply. We assume that the (real) price process B of the savings account has dynamics

$$\mathrm{d}B_t = r_t B_t \mathrm{d}t, \quad B_0 = 1, \tag{2.2}$$

where the *(real) interest rate* $(r_t)_{t\in[0,T]}$ satisfying $\int_0^T |r_t|\,\mathrm{d}t < \infty$ is to be determined in equilibrium. (It will always be deterministic below, but this is not needed for the formulation of the model.) We often use the savings account as numéraire; the corresponding quantities are called *nominal*.

The stock is a claim to an exogenous *(real) dividend rate process* δ relative to ν. (This means that the cumulative real dividends paid until time t are $\int_0^t \delta_u \nu(\mathrm{d}u)$.) We assume that δ is an Itô process driven by a P-Brownian motion W^δ. Moreover, we suppose that the (nominal, cum-dividend) stock price is an Itô process with continuous and nonvanishing variance, driven by the same Brownian motion W^δ:

$$\mathrm{d}S_t = \mu_t^S\,\mathrm{d}t + \sqrt{\Sigma_t^S}\,\mathrm{d}W_t^\delta, \quad S_0 \in \mathbb{R}_+. \tag{2.3}$$

Here, the initial stock price S_0, the drift rate $(\mu_t^S)_{t\in[0,T]}$, and the (instantaneous) variance $(\Sigma_t^S)_{t\in[0,T]}$ are to be determined in equilibrium.

At the initial time $t = 0$, agents start with zero savings and some position $\hat{\varphi}_0$ in the stock. (For equilibrium applications, the total initial allocations of course need to match the unit net supply of the stock.) *Trading strategies* are in turn described by predictable processes $\varphi \in L(S)$, where φ_t denotes the number of shares in the stock held at time $t > 0$.[11] To rule out doubling strategies, we assume that the set $\mathcal{M}^2(S)$ of equivalent martingale measures for S with square-integrable density is nonempty and focus on *admissible* strategies which satisfy[12]

$$\varphi \bullet S \text{ is a } Q\text{-supermartingale for all } Q \in \mathcal{M}^2(S). \tag{2.4}$$

In addition to the gains and losses generated by trading, agents receive a (nominal) cumulative endowment process Y, which is adapted and of finite variation. In most examples, it is derived from a (real) endowment *rate* process y relative to the consumption clock ν satisfying $\int_0^T |y_t|\,\nu(\mathrm{d}t) < \infty$. In this case, the (nominal) cumulative endowment is $Y_t = \int_0^t \frac{y_t}{B_t}\,\nu(\mathrm{d}t)$.

Agents use their endowments and gains from trading to finance (real) consumption rates c satisfying $E\left[\left(\int_0^T |c_t|\,\nu(\mathrm{d}t)\right)^2\right] < \infty$. (This means that the cumulative amount consumed until time t is $\int_0^t c_u \nu(\mathrm{d}u)$.) The (nominal) *wealth process* corresponding to an endowment Y, an admissible trading strategy φ, and a consumption rate c is

$$\mathrm{d}X_t^{\varphi,c,Y} = \varphi_t\,\mathrm{d}S_t - \frac{c_t}{B_t}\,\nu(\mathrm{d}t) + \mathrm{d}Y_t, \quad X_0^{\varphi,c,Y} = \hat{\varphi}_0 S_0. \tag{2.5}$$

A consumption rate c is called *feasible* for a given admissible strategy φ and an endowment Y if

$$X_T^{\varphi,c,Y} \geq 0. \tag{2.6}$$

The controls available to each agent are the *admissible portfolio/consumption pairs*, each of which

[11] As is customary, the corresponding dynamics of the savings account are implicitly determined by the self-financing condition (2.5).

[12] If the stock price is sufficiently integrable, this class includes bounded and other sufficiently integrable strategies. Similar notions of admissibility for utilities defined on the whole real line are used in [19, 63], for example.

are consisting of an admissible trading strategy and a corresponding feasible consumption rate. This set is denoted by $\mathcal{A}(\hat{\varphi}_0, Y)$.

2.4 Market with Frictions

To the above economy, we now add *proportional transaction costs* for purchases and sales of the stock. This means that when buying or selling $\mathrm{d}\|\varphi\|_t$ shares, agents have to pay a cost of $\varepsilon_t \, \mathrm{d}\|\varphi\|_t$, where the nonnegative, adapted transaction cost process $(\varepsilon_t)_{t\in[0,T]}$ describes the absolute size of the trading cost. This allows to cover as special cases two specifications typically used in the literature: transaction costs proportional to the amount of wealth transacted (cf. [49, 15] and many more recent studies), or transaction costs proportional to the number of shares bought or sold (e.g., [66, 36, 50]).

Admissibility with transaction costs is defined in direct analogy to the corresponding frictionless notion (2.4) by requiring that the (cost-adjusted) gains process is a supermartingale under all (square-integrable) equivalent martingale measures:

$$\varphi \bullet S - \varepsilon \bullet \|\varphi\| \text{ is a } Q\text{-supermartingale for all } Q \in \mathcal{M}^2(S). \tag{2.7}$$

We then also say that φ is ε-*admissible.*

Remark 2.1. For strictly positive transaction costs, the supermartingale condition (2.7) implies, in particular, that admissible trading strategies with transaction costs are a.s. of finite variation. Indeed, by Lemma E.5, a strategy φ satisfies (2.7) if and only if it satisfies the frictionless admissibility condition (2.4) and

$$E^Q\left[\int_{(0,T]} \varepsilon_t \, \mathrm{d}\|\varphi\|_t\right] < \infty \text{ for all } Q \in \mathcal{M}^2(S). \tag{2.8}$$

The (nominal) *frictional wealth process* corresponding to an admissible trading strategy φ, a consumption rate c, a cumulative endowment process Y, and a transaction cost process ε is

$$\mathrm{d}X_t^{\varphi,c,Y,\varepsilon} = \varphi_t \, \mathrm{d}S_t - \varepsilon_t \, \mathrm{d}\|\varphi\|_t - \frac{c_t}{B_t}\, \nu(\mathrm{d}t) + \mathrm{d}Y_t, \quad X_0^{\varphi,c,Y,\varepsilon} = \hat{\varphi}_0 S_0. \tag{2.9}$$

Again in direct analogy to the frictionless case (2.6), a consumption rate c is called *feasible* for a given admissible trading strategy φ, an endowment process Y, and transaction cost process ε if

$$X_T^{\varphi,c,Y,\varepsilon} \geq 0. \tag{2.10}$$

The set of admissible portfolio/consumption pairs with transaction cost process ε consists of admissible trading strategies and corresponding feasible consumption rates; it is denoted by $\mathcal{A}_\varepsilon(\hat{\varphi}_0, Y)$.

3 Individual Optimality

In this section, we consider the agents' individual optimization problems, taking a savings account B as in (2.2) and a cum-dividend stock price S as in (2.3) as exogenously given. These results are the building blocks for the general equilibrium results presented in Section 4. Throughout, we assume that the interest rate $(r_t)_{t\in[0,T]}$ is deterministic; more general specifications would require a wealth-dependent shift of the no-trade region corresponding to the optimal policy, compare the discussion in [42].

3.1 Preferences

We consider three agents $i = 1, 2, 3$ with constant absolute risk aversions $\gamma^i > 0$ and deterministic impatience rates $\beta^i(t)$. That is, their (time-dependent) utility functions are

$$U_t^i(x) = -\exp\left(-\int_0^t \beta^i(u)\,\mathrm{d}u\right)\exp\left(-\gamma^i x\right). \tag{3.1}$$

3.2 Agents' Individual Optimization Problems

Agents $i = 1, 2$ and Agent 3 play different roles in the economy. Whereas Agents 1 and 2 trade the stock to share their endowment risks, Agent 3 solves a pure consumption-savings problem. With transaction costs, Agent 3 receives the transaction cost payments made by Agents 1 and 2 and can therefore be interpreted as a government collecting taxes or the operator of an exchange receiving trading fees. By smoothing out the (singular) transaction cost payments, Agent 3 allows to clear the goods market for standard models where endowments and dividends arrive with finite rates. Note that matching this exogenous absolutely-continuous supply and Agent 1 and 2's (also absolutely continuous) consumption demand with the transaction cost payments directly is not possible for diffusive asset dynamics, because the transaction costs then are of local-time type (i.e., they are either zero or paid at an infinite rate).

This issue does not arise in the model of [48], where the savings account is exogenous and only the stock market has to clear. It is also absent from discrete-time models (where both in- and outflows are discrete) and from models with quadratic trading costs (where trading rates and in turn transaction costs are absolutely continuous). With *small* proportional transaction costs, the introduction of Agent 3 could also be avoided without affecting our main results by smoothing out Agent 1 and 2's trading strategies appropriately, compare Section 4.3. However, explicitly modeling the entity receiving the transaction costs allows to formulate a continuous-time notion of equilibrium with proportional transaction costs of arbitrary size. Moreover, modeling "where the transaction costs go", rather than letting them disappear from the model, gives the model a stronger general equilibrium flavor and opens the door to the welfare analysis of redistributive effects, for example.

Agents 1 and 2

Agents 1 and 2 receive (real) endowments at rates $(y_t^1)_{t\in[0,T]}$ and $(y_t^2)_{t\in[0,T]}$ and trade both the financial and the dividend-paying asset to share the corresponding risks.

They start at time $t = 0$ with zero savings and $\hat{\varphi}_0^1 \geq 0$ and $\hat{\varphi}_0^2 \geq 0$ shares of the stock, respectively, where $\hat{\varphi}_0^1 + \hat{\varphi}_0^2 = 1$ to ensure initial stock market clearing.

Frictionless optimization. In the frictionless economy of Section 2.3, Agent $i = 1, 2$ solves

$$E\left[\int_0^T U_t^i(c_t)\,\nu(\mathrm{d}t)\right] \to \max_{(\varphi,c)\in\mathcal{A}(\hat{\varphi}_0^i,Y^i)}!, \tag{3.2}$$

where $Y_t^i = \int_{(0,t]} \frac{y_t^i}{B_t}\,\nu(\mathrm{d}t)$ is the respective nominal cumulative endowment. The frictionless optimization problem (3.2) has been the subject of intensive research [27, 62, 19, 37, 53, 63, 5, 6, 54]. In the present study, a sufficiently regular solution is our point of departure for the analysis of an additional small transaction cost. Whence, we henceforth assume that for $i = 1, 2$ there exist:

i) an admissible portfolio/consumption pair $(\hat{\varphi}^i, \hat{c}^i)$, where $\hat{\varphi}^i$ is an Itô process with continuous and nonvanishing volatility, $E\left[\int_0^T |U_t^i(\hat{c}_t^i)|\,\nu(\mathrm{d}t)\right] < \infty$, and the budget constraint (2.6) is satisfied with equality;

ii) a positive square-integrable martingale $(\hat{Z}_t^i)_{t\in[0,T]}$ such that the corresponding marginal pricing measure $\hat{Q}^i \approx P$ on $\mathcal{F}_T$ given by $\mathrm{d}\hat{Q}^i = \frac{\hat{Z}_T^i}{\hat{Z}_0^i}\,\mathrm{d}P$ is an equivalent martingale measure for S;

such that the gains process $\hat{\varphi}^i \bullet S$ is a $\hat{Q}^i$-martingale and the consumption rate $\hat{c}^i$ is linked to the martingale $\hat{Z}^i$ by the following first-order condition:

$$U_t^{i\prime}(\hat{c}_t) = \frac{\hat{Z}_t^i}{B_t}, \quad t \in [0, T]. \tag{3.3}$$

These conditions imply the optimality of $(\hat{\varphi}^i, \hat{c}^i)$ for (3.2); see Lemma A.1. Their validity in a concrete model is verified in Section 5.

Frictional optimization. In the market with transaction costs described in Section 2.4, Agent $i \in \{1, 2\}$ solves

$$E\left[\int_0^T U_t^i(c_t)\,\nu(\mathrm{d}t)\right] \to \max_{(\varphi,c)\in\mathcal{A}_{\varepsilon^i}(\hat{\varphi}_0^i,Y^i)}!, \tag{3.4}$$

where $(\varepsilon_t^i)_{t\in[0,T]}$ is the proportional trading cost of agent i. We assume that

$$\varepsilon_t^i = \varepsilon\alpha^i\mathfrak{E}_t.$$

Here, the (small) parameter $\varepsilon > 0$ controls the size of the total transaction costs, whereas the equilibrium weights $\alpha^1, \alpha^2 \in (0, 1)$ with $\alpha^1 + \alpha^2 = 1$ describe how Agents 1 and 2 split the exogenous normalized transaction cost $(\mathfrak{E}_t)_{t\in[0,T]}$, a positive adapted Itô process with continuous variance. If agents' risk aversions are identical as in [48], each of them will pay half of the total trading cost in equilibrium, cf. Theorem 4.1. In general, a larger share will be allocated to the more risk-averse agent, who has a stronger motive to hedge her endowment risk.

Asymptotic optimality. The optimization problem (3.4) with transaction costs is intractable already in simple concrete models. As a way out, we look for portfolio/consumption pairs $(\hat{\varphi}^{1,\varepsilon}, \hat{c}^{1,\varepsilon})$ and $(\hat{\varphi}^{2,\varepsilon}, \hat{c}^{2,\varepsilon})$ that are *asymptotically optimal* for *small* transaction costs. This means that the policies should minimize the performance loss compared to the frictionless case at the leading asymptotic order as ε goes to zero, cf. Theorem 3.2.

Such asymptotically optimal policies have been determined for rather general asset price dynamics by [50, 65, 56, 61, 41, 42, 10]. Here, we extend these results to include the additional features required for a general equilibrium setting: intermediate consumption, non-constant interest rates, dividends, as well as random and time-varying transaction costs. Modulo stopping (needed to ensure that the candidate strategy is sufficiently integrable), the asymptotically optimal trading strategy corresponds to the minimal amount of trading necessary to remain in a random and time-dependent *no-trade region*

$$[\hat{\varphi}^i - \varepsilon^{\frac{1}{3}}\Delta\overline{\varphi}^i, \hat{\varphi}^i + \varepsilon^{\frac{1}{3}}\Delta\overline{\varphi}^i] \tag{3.5}$$

around the frictionless optimizer $\hat{\varphi}^i$. Here, the normalized halfwidth of the no-trade region is

$$\Delta\overline{\varphi}^i_t := \left(\frac{3}{2}\alpha^i \mathfrak{E}_t \frac{A_t}{\gamma^i} \frac{\Sigma^{\hat{\varphi}^i}_t}{\Sigma^S_t}\right)^{\frac{1}{3}}, \tag{3.6}$$

where the *annuity process* $(A_t)_{t\in[0,T]}$ (the nominal value of a unit payment stream) is given by

$$A_t = \int_{[t,T]} \frac{1}{B_s}\,\nu(\mathrm{d}s). \tag{3.7}$$

Like for the other primitives of the model, we assume that $\Delta\overline{\varphi}^i$ is an Itô process with continuous variance.[13] The existence of strategies $\hat{\varphi}^{i,\varepsilon}$, $i = 1, 2$, corresponding to the minimal amount of trading to remain in the no-trade regions (3.5) is then guaranteed by [1, Lemma 4.1.1]:

Proposition 3.1. *Let $i \in \{1, 2\}$. For each $\varepsilon > 0$, there exists a unique continuous adapted process $(\Delta\varphi^{i,\varepsilon})_{t\in[0,T]}$ null at 0 taking values in $[-\varepsilon^{\frac{1}{3}}\Delta\overline{\varphi}^i, \varepsilon^{\frac{1}{3}}\Delta\overline{\varphi}^i]$, which satisfies on $(0, T]$ the Skorokhod SDE*

$$\mathrm{d}\Delta\varphi^{i,\varepsilon}_t = -\,\mathrm{d}\hat{\varphi}^i_t$$

with instantaneous reflection at $\pm\varepsilon^{\frac{1}{3}}\Delta\overline{\varphi}^i$. (That is, $\Delta\varphi^{i,\varepsilon}$ has an "infinite inward-pointing drift" at $\pm\varepsilon^{\frac{1}{3}}\Delta\overline{\varphi}^i$ so that $\Delta\varphi^{i,\varepsilon}$ does not leave the interval $[-\varepsilon^{\frac{1}{3}}\Delta\overline{\varphi}^i, \varepsilon^{\frac{1}{3}}\Delta\overline{\varphi}^i]$.)

Set $\varphi^{i,\varepsilon} = \hat{\varphi}^i + \Delta\varphi^{i,\varepsilon}$ so that $\varphi^{i,\varepsilon}$ takes values in the no-trade region (3.5). An appropriately stopped version of $\varphi^{i,\varepsilon}$ is indeed asymptotically optimal for small transaction costs:[14]

Theorem 3.2. *Let $i \in \{1, 2\}$ and fix $\ell^i \in \mathbb{R}$. Suppose the interest rate $(r_t)_{t\in[0,T]}$ is deterministic and the integrability conditions in Assumption A.2 are satisfied for some $\kappa > 0$. Then, the*

[13] If $\mathfrak{E}^i$, $\Sigma^{\hat{\varphi}^i}$, and Σ^S are positive Itô processes, this follows from Itô's formula. Drift and quadratic variation of $\Delta\overline{\varphi}^i$ under $\hat{Q}^i$ can then be expressed in terms of their counterparts for $\mathfrak{E}$, $\Sigma^{\hat{\varphi}^i}$, and Σ^S. However, the integrability conditions required below are most conveniently expressed in terms of $\Delta\overline{\varphi}^i$.

[14] This extends the main result of [39] to random endowments, deterministic interest rates, intermediate consumption as well as random and time-varying transaction costs.

portfolio/consumption pair

$$\hat{\varphi}^{i,\varepsilon} := \varphi^{i,\varepsilon}\mathbf{1}_{[\![0,\hat{\tau}^i_\varepsilon]\!]} + \ell^i \mathbf{1}_{]\!]\hat{\tau}^i_\varepsilon,T]\!]},$$
$$\hat{c}^{i,\varepsilon} := \hat{c}^i - \frac{\varepsilon^i}{A} \bullet \|\hat{\varphi}^{i,\varepsilon}\| + \frac{\hat{\varphi}^{i,\varepsilon} - \hat{\varphi}^i}{A} \bullet S,$$

is admissible and optimal for (3.4) *at the leading-order* $O(\varepsilon^{\frac{2}{3}})$*:*

$$E\left[\int_0^T U_t^i(\hat{c}_t^{i,\varepsilon})\,\nu(\mathrm{d}t)\right] = E\left[\int_0^T U_t^i(\hat{c}_t^i)\,\nu(\mathrm{d}t)\right] - \frac{3}{4}\varepsilon^{\frac{2}{3}}\hat{Z}_0^i E^{\hat{Q}^i}\left[\int_0^T \frac{\alpha^i \mathfrak{E}_t}{\Delta\overline{\varphi}_t^i}\Sigma_t^{\hat{\varphi}^i}\,\mathrm{d}t\right] + o(\varepsilon^{\frac{2}{3}}) \tag{3.8}$$

$$= \sup_{(\varphi^\varepsilon, c^\varepsilon)\in\mathcal{A}_{\varepsilon^i}(\hat{\varphi}_0^i, Y^i)} E\left[\int_0^T U_t^i(c_t^\varepsilon)\,\nu(\mathrm{d}t)\right] + o(\varepsilon^{\frac{2}{3}}). \tag{3.9}$$

Here, the family $(\hat{\tau}^i_\varepsilon)_{\varepsilon>0}$ *of stopping times defined in* (A.23) *satisfies* $\hat{Q}^i[\tau^i_\varepsilon < T] = o(\varepsilon^{\frac{2}{3}})$*.*

Remark 3.3. The strategy $\hat{\varphi}^{i,\varepsilon}$ coincides with the reflected process $\varphi^{i,\varepsilon}$ until time $\hat{\tau}^i_\varepsilon$, when it is terminated to the static position ℓ^i. For individual optimality in partial equilibrium models, liquidation of the stock position ($\ell^i = 0$) is the canonical choice – this corresponds to closing out the portfolio to limit excessive losses. In general equilibrium models, the liquidation portfolios need to add up to the unit net supply of the stock, i.e., $\ell^1 + \ell^2 = 1$. As the transaction costs become small, the probability of this liquidation becomes negligible.

Agent 3

We now turn to Agent 3, who does not trade the stock. Starting from zero savings and given an exogenous endowment rate $(y_t^3)_{t\in[0,T]}$, Agent 3 solves a pure consumption-savings problem.

Frictionless optimization. In the frictionless baseline model, Agent 3's optimization problem is

$$E\left[\int_0^T U_t^3(c_t)\,\nu(\mathrm{d}t)\right] \to \max_{(0,c)\in\mathcal{A}(0,Y^3)}!, \tag{3.10}$$

where $Y_t^3 = \int_{(0,t]} \frac{y_t^3}{B_t}\,\nu(\mathrm{d}t)$. Like for Agent 1 and 2, we assume that there exist:

i) a feasible consumption rate $\hat{c}^3$ with $E\left[\int_0^T |U_t^3(\hat{c}_t^3)|\,\nu(\mathrm{d}t)\right] < \infty$ such that the budget constraint (2.10) (with $\varphi \equiv 0$) is satisfied with equality;

ii) a positive square-integrable martingale $(\hat{Z}_t^3)_{t\in[0,T]}$ with corresponding marginal pricing measure $\hat{Q}^3 \approx P$ on $\mathcal{F}_T$ given by $\mathrm{d}\hat{Q}^i = \frac{\hat{Z}_T^i}{\hat{Z}_0^i}\,\mathrm{d}P$;

such that

$$U_t^{3\prime}(\hat{c}_t^3) = \frac{\hat{Z}_t^3}{B_t}, \quad t\in[0,T]. \tag{3.11}$$

This first-order condition implies that $\hat{c}^3$ is optimal for (3.10); cf. [34, Lemma 3.1]. Its validity in a concrete model is verified in Section 5.

Frictional optimization. In the economy with transaction costs, Agent 3 receives the transaction costs paid by Agents 1 and 2 in addition to her baseline endowment Y^3. If Agents 1 and 2 use the asymptotically optimal strategies $\hat{\varphi}^{1,\varepsilon}$ and $\hat{\varphi}^{2,\varepsilon}$ from Theorem 3.2, Agent 3's total endowment in turn is

$$Y^3 + \Delta Y^\varepsilon, \quad \text{where } \Delta Y^\varepsilon := \varepsilon^1 \bullet \|\hat{\varphi}^{1,\varepsilon}\| + \varepsilon^2 \bullet \|\hat{\varphi}^{2,\varepsilon}\|.$$

Her consumption-savings problem then reads as follows:

$$E\left[\int_0^T U_t^3(c_t)\,\nu(\mathrm{d}t)\right] \to \max_{(0,c)\in\mathcal{A}(0,Y^3+\Delta Y^\varepsilon)}! . \tag{3.12}$$

Theorem 3.4. *Suppose the assumptions of Theorem 3.2 are satisfied and, in addition, the integrability conditions in Assumption A.3 hold.*[15] *Then, the consumption rate*

$$\hat{c}^{3,\varepsilon} = \hat{c}^3 + \frac{\varepsilon^1}{A} \bullet \|\hat{\varphi}^{1,\varepsilon}\| + \frac{\varepsilon^2}{A} \bullet \|\hat{\varphi}^{2,\varepsilon}\|$$

is feasible and optimal at the leading order $O(\varepsilon^{\frac{2}{3}})$ for (3.12)*:*

$$\begin{aligned} E\left[\int_0^T U_t^3(\hat{c}_t^{3,\varepsilon})\,\nu(\mathrm{d}t)\right] &= E\left[\int_0^T U_t^3(\hat{c}_t^3)\,\nu(\mathrm{d}t)\right] \\ &\quad + \frac{1}{2}\varepsilon^{\frac{2}{3}}\hat{Z}_0^3 E^{\hat{Q}^3}\left[\int_0^T \mathfrak{E}_t\left(\alpha^1\frac{\Sigma_t^{\hat{\varphi}^1}}{\Delta\overline{\varphi}_t^1} + \alpha^2\frac{\Sigma_t^{\hat{\varphi}^2}}{\Delta\overline{\varphi}_t^2}\right)\mathrm{d}t\right] + o(\varepsilon^{\frac{2}{3}}) &(3.13)\\ &\geq \sup_{(0,c)\in\mathcal{A}(0,Y^3+\Delta Y^\varepsilon)} E\left[\int_0^T U_t^3(c_t^\varepsilon)\,\nu(\mathrm{d}t)\right] + o(\varepsilon^{\frac{2}{3}}). &(3.14)\end{aligned}$$

4 Equilibrium

In Section 3 we have considered the agents' individual optimization problems, taking the asset price dynamics as exogenously given. Now, we extend this analysis to a general equilibrium setting, where asset prices are determined endogenously by matching supply and demand. As in Section 3, our starting point is a sufficiently regular frictionless baseline model, which we then perturb with a small proportional transaction cost.

4.1 Frictionless Equilibrium

In the frictionless market from Section 2.3 we use the standard notion of equilibrium [58, 22, 44]. To wit, a *(Radner) equilibrium* is a tuple $(r, S_0, \mu^S, \Sigma^S, \hat{\varphi}^1, \hat{c}^1, \hat{\varphi}^2, \hat{c}^2, \hat{c}^3)$ for which the frictionless market with real interest rate r and cum-dividend stock price (2.3) with initial value S_0, drift μ^S, and (continuous and positive) instantaneous variance Σ^S satisfies:

i) $(\hat{\varphi}^i, \hat{c}^i) \in \mathcal{A}(\hat{\varphi}_0^i, Y^i)$ is optimal for Agent $i = 1, 2$'s optimization problem (3.2), and $(0, \hat{c}^3) \in \mathcal{A}(0, Y^3)$ is optimal for Agent 3's optimization problem (3.10).

ii) $\hat{\varphi}^1 + \hat{\varphi}^2 = 1$, i.e., the stock market clears.

[15] In particular, the interest rate is deterministic.

iii) $\hat{c}^1 + \hat{c}^2 + \hat{c}^3 = \delta + y^1 + y^2 + y^3$, i.e., the goods market clears.

Throughout, we assume that a frictionless equilibrium exists, which satisfies the assumptions for the agents' individual optimization problems made in Sections 3.2. This is verified for a concrete model in Section 5.

4.2 Equilibrium with Friction

We now turn to Radner equilibria with transaction costs. As in Section 3, this means that whenever Agents 1 and 2 trade the stock, together they have to pay a *total* proportional transaction cost $\varepsilon_t = \varepsilon \mathfrak{C}_t$, where the (small) parameter ε controls the size of the cost, whereas its dynamics are described by the Itô process $\mathfrak{C}$. Both of these quantities are exogenous; in contrast, we endogenously determine how the transaction cost is split between the agents in equilibrium, where the more risk averse agent has a stronger motive to trade and is therefore willing to bear higher trading costs. (If all agents have the same risk aversion as in Lo et al. [48], then each agent pays approximately half of the cost both in their and in our model, compare [48, Equation (32b)] and Theorem 4.1, respectively.)

Accordingly, an *asymptotic (Radner) equilibrium with small transaction costs* is defined as a tuple $(r^\varepsilon, S_0^\varepsilon, \mu^{S,\varepsilon}, \Sigma^{S,\varepsilon}, \alpha^1, \alpha^2, \hat{\varphi}^{1,\varepsilon}, \hat{c}^{1,\varepsilon}, \hat{\varphi}^{2,\varepsilon}, \hat{c}^{2,\varepsilon}, c^{3,\varepsilon})$ such that the frictional market with real interest rate r^ε, cum-dividend stock price (2.3) with initial value S_0^ε, drift $\mu^{S,\varepsilon}$ and (continuous and positive) instantaneous variance $\Sigma^{S,\varepsilon}$, and transaction costs $(\varepsilon_t^i)_{t\in[0,T]} = (\varepsilon\alpha_i \mathfrak{C}_t)_{t\in[0,T]}$ for Agent $i \in \{1,2\}$, satisfies:

i) $(\hat{\varphi}^{i,\varepsilon}, \hat{c}^{i,\varepsilon^i}) \in \mathcal{A}_{\varepsilon^i}(\hat{\varphi}_0^i, Y^i)$ is *asymptotically* optimal for Agent $i = 1, 2$'s optimization problem (3.4), and $(0, \hat{c}^{3,\varepsilon}) \in \mathcal{A}(0, Y^3 + \Delta Y^\varepsilon)$ is *asymptotically* optimal for Agent 3's optimization problem (3.12).

ii) $\hat{\varphi}^1 + \hat{\varphi}^2 = 1$, i.e., the stock market clears.

iii) $\hat{c}^1 + \hat{c}^2 + \hat{c}^3 = \delta + y^1 + y^2 + y^3$, i.e., the goods market clears.

iv) $\alpha^1 + \alpha^2 = 1$, i.e., the total transaction costs are split between Agents 1 and 2.

Note that this notion of equilibrium is *asymptotic* in the sense that we only require the agents' policies to be asymptotically optimal for small transaction costs, similar to the concept of ε-equilibria in game theory, cf. [59] and many more recent studies. In contrast, market clearing is enforced *exactly*.

We are now ready to state our main stability result. It shows that if the agents split the transaction costs appropriately, then the *frictionless* equilibrium prices still clear the market:

Theorem 4.1. *Suppose there exists a frictionless equilibrium* $(r, S_0, \mu^S, \Sigma^S, \hat{\varphi}^1, \hat{c}^1, \hat{\varphi}^2, \hat{c}^2, \hat{c}^3)$ *in the sense of Section 4.1, for which the assumptions of Theorems 3.2 and 3.4 are satisfied.*[16] *Then, if the transaction costs are split according to Agent 1 and 2's shares of the total risk aversion,*

$$\alpha^1 = \frac{\gamma^1}{\gamma^1 + \gamma^2} \quad \textit{and} \quad \alpha^2 = \frac{\gamma^1}{\gamma^1 + \gamma^2}, \tag{4.1}$$

[16] In particular, the interest rate has to be deterministic.

the frictionless equilibrium prices also support an asymptotic equilibrium with small transaction costs. To wit, with $r^\varepsilon = r$, $S_0^\varepsilon = S_0$, $\mu^{S,\varepsilon} = \mu^S$, $\Sigma^{S,\varepsilon} = \Sigma^S$, *and the corresponding asymptotically optimal policies* $(\hat{\varphi}^{1,\varepsilon}, \hat{c}^{1,\varepsilon})$ *and* $(\hat{\varphi}^{2,\varepsilon}, \hat{c}^{2,\varepsilon})$ *from Theorem 3.2 (with* $\ell^1 = \ell^2 = \frac{1}{2}$*), and* $\hat{c}^{3,\varepsilon}$ *from Theorem 3.4, the tuple* $(r^\varepsilon, S_0^\varepsilon, \mu^{S,\varepsilon}, \Sigma^{S,\varepsilon}, \alpha^1, \alpha^2, \hat{\varphi}^{1,\varepsilon}, \hat{c}^{1,\varepsilon}, \hat{\varphi}^{2,\varepsilon}, \hat{c}^{2,\varepsilon}, \hat{c}^{3,\varepsilon})$ *is an asymptotic equilibrium with small transaction costs.*[17]

4.3 Robustness checks

In this section, we explore several modifications of our model and discuss to what extent they affect the validity of Theorem 4.1.

Redistribution of the transaction costs. In our model, the entity receiving the transaction costs (Agent 3) is modeled explicitly for two reasons. On the one hand, by smoothing the singular transaction costs payments into an absolutely continuous consumption stream, this additional agent allows to clear the goods market in standard models where endowments and dividends are paid at finite rates. On the other hand, we think that modeling "where the transaction costs go" gives the model a stronger general equilibrium flavor then simply letting them disappear outside the model.

However, for the validity of Theorem 4.1 in the context of asymptotically *small* transaction costs the presence of the third agent is not crucial. Indeed, one can replace Agent 1 and 2's singular continuous strategies (which do not trade in the interior of the respective no-trade regions, but at an infinite rate once their boundaries are breached) by suitably smoothed approximations. As the transaction costs tend to zero, the performance of these approximating strategies becomes indistinguishable from the policies in Theorem 3.2.

Other trading costs. Another important variation of the model concerns the nature of the exogenous trading costs. We have chosen to work with proportional costs because most proposals and implementations of financial transaction taxes are of this form. However, modulo technical details, the main message of Theorem 4.1 remains valid much more broadly.

This is most easily seen for fixed transaction costs as considered by Lo et al. [48]. In this case, the optimal policies again correspond to no-trade regions, where the present position is simply held until the boundaries are breached. When this happens, fixed costs make it optimal to rebalance all the way back to the frictionless target portfolio, rather than just performing the minimal amount of trading to remain in the no-trade region as is optimal with proportional costs. Despite this technical difference (which leads to an impulse rather than a singular control problem), the respective asymptotically optimal policies are of a very similar form. Indeed, Formula (3.6) for the optimal halfwidth remains valid, up to a change of the power and constant [46, 48, 61, 4, 10, 11, 26]. An inspection of the proof of Theorem 4.1 in turn shows that – with the same split of the transaction costs according to agents' risk aversions – the frictionless equilibrium prices also still clear the market with small fixed costs.

Mutatis mutandis, the same remains true with quadratic costs levied on the turnover rate of the portfolio [2, 28, 29, 3, 31, 52]. In this case, there is no no-trade region; instead, agents always

[17] Note that the liquidation times $\hat{\tau}_\varepsilon^1$ and $\hat{\tau}_\varepsilon^2$ of Agents 1 and 2, defined in (A.23), coincide.

trade towards the frictionless target portfolio with a finite, absolutely continuous rate. To achieve stock market clearing, these trading rates have to match for Agents 1 and 2. The explicit formula from [52] shows that this is once more satisfied if and only if each agent's share of the total trading cost is given by (4.1). The remaining arguments for market clearing can then be carried through verbatim as in the proof of Theorem 4.1.

In summary, these results suggest that the validity of Theorem 4.1 is not intimately linked to the precise nature of the trading cost.

Non-diffusive prices. The asymptotic analysis of the agents' individual optimizations problems crucially exploits that asset prices and endowments are driven by diffusion processes, so that the frictionless optimal strategies generically have nontrivial Brownian components as well. For example, it is a consequence of elementary scaling properties of Brownian motion that the leading-order welfare effect of small proportional costs is of order $O(\varepsilon^{\frac{2}{3}})$, cf. [35, 60]. If the primitives of the frictionless model are driven by processes whose fluctuations scale differently, then both the asymptotic orders and the constants determining the optimal trading policies change. The intuition is that smoother target strategies can be tracked with less rebalancing, so that the optimal no-trade regions are narrower and the corresponding welfare effect is smaller.

However, the results of Rosenbaum and Tankov [61, Proposition 3] suggest that also in such more general settings, Theorem 4.1 still applies as long as infinitesimally small up and down movements are equally likely for the driving processes.[18] As a consequence, the fine structure of the underlying processes also does not play an overly important role.

More general utilities. Whereas Theorem 4.1 is robust with respect to the variations of the model discussed so far, the assumption that all agents have constant absolute risk aversion plays a crucial role. Indeed, if risk aversions are not constant, the wealth effects of transaction costs alter the target portfolio, which no longer coincides with the frictionless optimizer in this case. To wit, with transaction costs wealth is typically lower, so that investors with decreasing absolute risk aversion typically reduce their demand for the risky asset. This in turn leads to a downward shift of the no-trade region for both Agents 1 and 2,[19] cf. [42, Equation (3.2)]. As a result, stock market clearing can no longer be achieved by retaining the frictionless equilibrium prices and splitting the trading costs so as to match the halfwidths of the respective no-trade regions.

The wealth effect evidently disappears if transaction costs are immediately refunded as in [17], and if agents do not internalize this then the asymptotically optimal no-trade regions remain unchanged (up to inserting the current *indirect* risk aversion at each time and state [42, Equation (3.2)]). In a numerical study of a discrete-time model populated by agents with constant *relative* risk aversion [9], "the refund leads to almost no change in the choices of individual investors". This suggests that also in our model, disregarding the shift of the midpoint of the asymptotically optimal no-trade region should not lead to big quantitative effects that invalidate the broad conclusions of Theorem 4.1.

[18] This means that the corresponding jump measures, which describe the intensities with which jumps of various sizes occur, are symmetric for small jumps.

[19] E.g., with power utilities, the target *fractions* still coincide with their frictionless counterparts [42, Section 4], but the corresponding target *number* of shares is lower if wealth is reduced due to transaction costs.

Exogenous split of the trading costs. Theorem 4.1 states that the frictionless equilibrium prices still clear the market *if* the transaction costs are split between Agent 1 and 2 according to their risk aversions. This corresponds to a reduced-form model for a bargaining mechanism in which the agent with the stronger motive to trade bears a larger share of the exogenous cost. If this mechanism is replaced by an equal split of the transaction costs, for example, then Theorem 4.1 only remains valid if the risk aversions of Agent 1 and 2 match as in the model of Lo et al. [48]. In contrast, for heterogenous risk aversions, an adjustment to the frictionless equilibrium prices (e.g., a liquidity premium that depends on the agents' current positions and their dynamics) is required. This is an important direction for future research, that is initiated in [7] in a simpler version of the present model, where trading costs and preferences are quadratic and only expected returns (but not volatilities and interest rates) are determined in equilibrium.

5 A Frictionless Benchmark Economy

In this section, we consider a concrete example for our main result, Theorem 4.1. This serves two purposes. On the one hand, the resulting closed-form frictionless equilibrium dynamics allow to shed further light on the interdependencies between volatility, small transaction costs, and trading volume. On the other hand, the example illustrates that all our technical assumptions can indeed be verified in a specific setting.

The main assumptions for Theorem 4.1 are that i) frictionless equilibrium asset prices and trading strategies have to be diffusions with nontrivial quadratic variations, and ii) frictionless equilibrium interest rates need to be deterministic. These assumptions are not met by the models hitherto considered in the literature. For example, the equilibrium asset price of [48] is constant; conversely, the trading strategies in [66, 67, 14] are continuous deterministic functions. The setting closest to our requirements is put forward in [13]. In their model, both equilibrium prices and strategies are diffusions, but the corresponding interest rates are stochastic.

Building on these studies, we now introduce a frictionless benchmark model that leads to deterministic interest rates, but stochastic and time-varying trading strategies and asset prices.

5.1 Exogenous Inputs

To define the exogenous inputs of the model, suppose that $(\Omega, \mathcal{F}, \mathbb{F} = (\mathcal{F}_t)_{t\in[0,T]}, P)$ supports four independent Brownian motions $(W_t^\delta, W_t^\xi, W_t^1, W_t^2)_{t\in[0,T]}$.

We assume that the consumption clock is given by $\nu(\,\mathrm{d}t) = \mathbf{1}_{(0,T)}\,\mathrm{d}t + \delta_T$.

As in [14], the (real) dividend *rate* follows an (inhomogeneous) Brownian motion with drift:[20]

$$\mathrm{d}\delta_t = \mu^\delta(t)\,\mathrm{d}t + \sqrt{\Sigma^\delta(t)}\,\mathrm{d}W_t^\delta, \quad \delta_0 \in \mathbb{R}_+, \tag{5.1}$$

for a continuous deterministic function $\mu^\delta(t)$ and a deterministic continuously differentiable function $\Sigma^\delta(t)$, where $\Sigma^\delta > 0$.

[20] As in [14], this specification leads to a diffusive equilibrium price. In contrast, a constant price obtains if the *cumulative* dividends follows Brownian motion as in [48].

Agent $i = 1, 2$'s (real) endowment rates have dynamics

$$\mathrm{d}y_t^i = \mu^{y^i}(t)\,\mathrm{d}t + \sqrt{\Sigma^{y^i}(t)}\,\mathrm{d}W_t^i + \xi_t^i\,\mathrm{d}W_t^\delta, \quad y_0^i \in \mathbb{R}_+. \tag{5.2}$$

Here, the drift rate $\mu^{y^i}(t)$ and the variance $\Sigma^{y^i}(t) \geq 0$ of the *unspanned* endowment are also continuous deterministic functions.[21] In contrast, the volatility ξ_t^i of the part of the endowment that will turn out to be *spanned* by the dividend-paying stock is stochastic.[22] Like the dividend rate, it follows an (inhomogeneous) Brownian motion with drift:

$$\mathrm{d}\xi_t^i = \mu^{\xi^i}(t)\,\mathrm{d}t + \sigma^{\xi^i}(t)\,\mathrm{d}W_t^\xi, \quad \xi_0^i \in \mathbb{R}_+, \tag{5.3}$$

for deterministic continuously differentiable functions $\mu^{\xi^i}(t)$ and $\sigma^{\xi^i}(t)$. Crucially, we assume that the aggregate volatility-of-volatility of the spanned endowment vanishes:

$$\sigma^{\xi^1}(t) + \sigma^{\xi^2}(t) = 0, \quad t \in [0, T]. \tag{5.4}$$

If this assumption is not satisfied, the equilibrium interest rate cannot be deterministic, compare [13]. This requirement is met, in particular, if the aggregate spanned endowment is zero like in [48]. For the applicability of the stability results, we also need to assume that $\sigma^{\xi^i} \neq 0$ (this will ensure that stock positions are diffusive) and that $|\sigma^{\xi^i}(0)| + \int_0^T |\sigma^{\xi^i\prime}(u)|\,\mathrm{d}u$ is sufficiently small (to ensure enough integrability, cf. Appendix C.3).

The endowment rate of Agent 3 (who receives the transaction cost payments) is for simplicity assumed to be constant, $y_t^3 = y_0^3 \in \mathbb{R}$.

Finally, we assume for simplicity that the normalized transactions costs are constant, $\mathfrak{C} = 1$.

5.2 Equilibrium Outputs

We now describe the frictionless equilibrium in the above model; for better readability, the derivation of these results is deferred to Appendix C.

First, the equilibrium (nominal) market price of risk λ^S for the stock is deterministic and given by[23]

$$\lambda^S(t) = \frac{\sqrt{\Sigma^\delta(t)} + \sum_{i=1}^2 \xi_t^i}{\frac{1}{\gamma^1} + \frac{1}{\gamma^2}}. \tag{5.5}$$

Second, the (real) equilibrium interest rate r is also deterministic. It is the unique solution to

[21] As in [13], stochastic volatility of the unspanned endowment would lead to stochastic interest rates.

[22] This specification is motivated by [48]; it leads to diffusive trading strategies. In contrast, if the spanned endowment volatility is *zero* as in [14], then the corresponding optimal strategies are deterministic.

[23] Note that even though ξ^1 and ξ^2 are stochastic, their sum is not, cf. (5.4).

the following integral equation:[24]

$$r(t) = \frac{1}{\sum_{i=1}^{3} \frac{1}{\gamma^i}} \left[\sum_{i=1}^{3} \frac{\beta^i(t)}{\gamma^i} + \mu^\delta(t) + \sum_{i=1}^{2} \mu^{y^i}(t) - \frac{1}{2} \sum_{i=1}^{2} \frac{1}{\gamma^i} \left(\lambda^S(t)\right)^2 - \frac{1}{2} \sum_{i=1}^{2} \gamma^i \Sigma^{y^i}(t) \right. \\ \left. - \frac{1}{2} \sum_{i=1}^{2} \gamma^i \sigma^{\xi^i}(t)^2 \left(\int_t^T \frac{A(u)}{A(t)} \lambda^S(u)\,\mathrm{d}u \right)^2 \right], \tag{5.6}$$

where $A(t) = \int_{[t,T]} \exp(-\int_0^u r(v)\,\mathrm{d}v)\,\nu(\mathrm{d}u)$ denotes the annuity process.

Third, like in most tractable equilibrium models (e.g., [23, 14]), the (nominal) equilibrium stock variance coincides with the (real) dividend variance up to appropriate discounting and therefore is also deterministic:

$$\Sigma^S(t) = A(t)^2 \Sigma^\delta(t). \tag{5.7}$$

The (nominal) drift rate of the stock is in turn given by another deterministic function:

$$\mu^S(t) = \lambda^S(t) A(t) \sqrt{\Sigma^\delta(t)}. \tag{5.8}$$

The corresponding initial stock price is a lengthy but explicit expression of the model parameters and the annuity process A, see (C.3).

Fourth, the optimal trading strategies of Agent $i = 1, 2$ in this economy are

$$\hat{\varphi}^i_t = \begin{cases} \frac{1}{\sqrt{\Sigma^\delta(t)}} \left(\frac{1}{\gamma^i} \lambda^S(t) - \xi^i_t \right), & \text{if } t > 0, \\ \hat{\varphi}^i_0, & \text{if } t = 0. \end{cases} \tag{5.9}$$

Evidently, these stock positions are diffusive precisely if the spanned endowment volatilities-of-volatilities ξ^i_t are stochastic processes with nontrivial quadratic variation. Moreover note that due to (5.4), we have $\langle \hat{\varphi}^1 \rangle = \langle \hat{\varphi}^2 \rangle$. The corresponding optimal consumption rates of Agents $i = 1, 2$ are Itô processes with dynamics

$$\mathrm{d}\hat{c}^i_t = -\frac{1}{\gamma^i} \left(\mathrm{d}M^i_t - \frac{1}{2} \mathrm{d}\langle M^i \rangle_t + \left(\beta^i(t) - r(t)\right) \mathrm{d}t \right), \tag{5.10}$$

where $\hat{c}^i_0$ is a lengthy but explicit expression of the model parameters and the annuity process A, see (C.4), and M^i is a martingale driven by W^d, W^i, and W^ξ_i with deterministic and bounded integrands which involve the model parameters and A, see (C.1).

Finally, the optimal consumption rate of Agent 3 is deterministic and satisfies

$$d\hat{c}^3(t) = -\frac{1}{\gamma^3} \left(\beta^3(t) - r(t)\right) \mathrm{d}t, \tag{5.11}$$

where $\hat{c}^3_0$ is an explicit expression of the model parameters and the annuity process A, see (C.5).

In summary, the exogenous inputs from Section 5.1 lead to a frictionless equilibrium where both the stock price and the agents' stock positions are stochastic processes with nontrivial quadratic

[24]Existence and uniqueness are established in Lemma D.1. If $\sigma^{\xi^i}_t = 0$ so that the spanned endowment volatilities are deterministic as in [14], then the last term in the second factor of (5.6) disappears, and a closed-form solution obtains.

variations. Nevertheless, the corresponding interest rate remains deterministic, as required for the application of Theorem 4.1.

5.3 Effects of Small Transaction Costs

Let us now discuss how the above equilibrium changes with the introduction of small proportional transaction costs ε. By Theorem 4.1, if the fraction of the cost paid by Agent i is her share of the total risk aversion, $\alpha^i = \gamma^i/(\gamma^1+\gamma^2)$, then the trading strategies corresponding to the no-trade regions with halfwidth (the equality is obtained by inserting (5.7) and (5.9))

$$\Delta \mathrm{NT}_t^\varepsilon = \left(\frac{3\varepsilon A_t}{2(\gamma^1+\gamma^2)}\frac{d\langle\hat{\varphi}^1\rangle_t}{d\langle S\rangle_t}\right)^{\frac{1}{3}} = \left(\frac{3}{2}\frac{\varepsilon}{A_t(\gamma^1+\gamma^2)}\frac{\sigma^{\xi_1}(t)^2}{\Sigma^\delta(t)^2}\right)^{\frac{1}{3}} \tag{5.12}$$

are asymptotically optimal (up to stopping) and clear the market. This formula is very similar to the leading-order optimal no-trade region of Lo et al. [48, Theorem 4]. The only changes concern powers and constants,[25] and the different way through which interest rates enter.[26]

Using Lemma A.9, the leading-order trading volume corresponding to the policies (5.12) can be readily computed in closed form as

$$||\hat{\varphi}_T^1|| + ||\hat{\varphi}_T^2|| = \int_0^T \left(\frac{128}{3}\frac{(\gamma^1+\gamma^2)A_t}{\varepsilon}\frac{\sigma^{\xi_1}(t)^4}{\Sigma^\delta(t)}\right)^{\frac{1}{3}} dt + o(\varepsilon^{-\frac{1}{3}}),$$

where the convergence is to be understood in probability. Again, up to a change of powers and constant, and a different role of the interest rate, this formula displays the same comparative statics as the leading-order result of [48].[27]

A Proofs for Individual Optimality

This section contains the proofs of Theorems 3.2 and 3.4 concerning asymptotically optimal policies with small transaction costs for individual investors. To this end, we compare an explicit candidate strategy/consumption pair (Theorem 3.2) or candidate consumption rate (Theorem 3.4) with an explicit dual upper bound. Such a convex duality approach for asymptotic verification was first used in a Mathematical Finance context by [33], and has recently been brought to bear on models with small transaction costs in [39, 1]. Here, we extend their analysis to random and time-varying transactions costs, intermediate consumption, dividends, and non-constant interest rates (as required for general equilibrium modeling). In doing so, we also weaken the required integrability conditions on the primitives of the model.

[25] This difference is due to the different transaction costs used, which are proportional in our model but fixed in [48]. Compare [4] for a detailed discussion.

[26] This difference arises because the equilibrium price in [48] is constant, whereas it is diffusive in our model.

[27] To obtain a formula for leading-order trading volume in their model, multiply the inverse of the approximate waiting time [48, Equation (30)] between successive trades with the corresponding trade size, which is given by the halfwidth of the no-trade region from [48, Theorem 4].

A.1 A Sufficient Condition for Frictionless Optimality

For the convenience of the reader, we first recall the well-known first-order condition for frictionless optimality:

Lemma A.1. *Fix $\hat{\varphi}_0 \in \mathbb{R}$ and let $(Y_t)_{t\in[0,T]}$ be a cumulative endowment process. If there are $(\hat{\varphi}, \hat{c}) \in \mathcal{A}^*(\hat{\varphi}_0, Y)$ with $E\left[\int_0^T |U_t(\hat{c}_t)|\,\nu(\mathrm{d}t)\right] < \infty$ and a positive square-integrable martingale $(\hat{Z}_t)_{t\in[0,T]}$ such that the density $\mathrm{d}\hat{Q} = \frac{\hat{Z}_T}{\hat{Z}_0}\,\mathrm{d}P$ defines an equivalent martingale measure for S, $\hat{\varphi} \bullet S$ is a $\hat{Q}$-martingale, and*

$$U_t'(\hat{c}_t) = \frac{\hat{Z}_t}{B_t}, \quad t \in [0,T], \tag{A.1}$$

then $(\hat{\varphi}, \hat{c})$ is optimal for (3.2).

Proof. Let $(\varphi, c) \in \mathcal{A}(\hat{\varphi}_0, Y)$ with $E\left[\int_0^T U_t^-(c_t)\,\nu(\mathrm{d}t)\right] < \infty$ be a competing strategy/consumption pair. Then using successively concavity of U, the first-order condition (A.1), Bayes' theorem as in [34, Corollary A.2(a)] (this requires the square integrability of $\hat{Z}$ since consumptions streams are typically not nonnegative here), the budget constraint (2.6) for c and $\hat{c}$ (which is binding for $\hat{c}$), the $\hat{Q}$-supermartingale property for $\varphi \bullet S$ and the $\hat{Q}$-martingale property for $\hat{\varphi} \bullet S$ give

$$\begin{aligned}
E\left[\int_0^T (U_t(c_t) - U_t(\hat{c}_t))\,\nu(\mathrm{d}t)\right] &\leq E\left[\int_0^T U_t'(\hat{c}_t)(c_t - \hat{c}_t)\,\nu(\mathrm{d}t)\right] \\
&= E\left[\int_0^T \hat{Z}_t\left(\frac{c_t}{B_t} - \frac{\hat{c}_t}{B_t}\right)\nu(\mathrm{d}t)\right] \\
&= \hat{Z}_0 E^{\hat{Q}}\left[\int_0^T \left(\frac{c_t}{B_t} - \frac{\hat{c}_t}{B_t}\right)\nu(\mathrm{d}t)\right] \\
&\leq \hat{Z}_0 E^{\hat{Q}}\left[\varphi \bullet S_T + Y_T - (\hat{\varphi} \bullet S_T + Y_T)\right] \\
&= \hat{Z}_0 E^{\hat{Q}}\left[\varphi \bullet S_T - \hat{\varphi} \bullet S_T\right] \leq 0.
\end{aligned}$$

Hence, $(\hat{\varphi}, \hat{c})$ is optimal as claimed. □

A.2 Assumptions on the Primitives

To control the remainder terms in the asymptotic expansions of Theorems 3.2 and 3.4, we need to assume that the primitives of the models are sufficiently integrable. The validity of these technical regularity conditions is verified for the concrete model of Section 5 in Appendix C.3. To formulate these assumptions succinctly, we introduce – for $i \in \{1, 2, 3\}$ and a generic measure $Q \approx P$ – the following sets:

$$
\begin{aligned}
\mathcal{X}_N^i(Q) &:= \left\{ \left(\frac{\mu^{\hat{\varphi}^i,Q} \Delta\overline{\varphi}^i}{\Sigma^{\hat{\varphi}^i}} \right)_T^*, \left(\frac{\mu^{\Delta\overline{\varphi}^i,Q} \Delta\overline{\varphi}^i}{\Sigma^{\hat{\varphi}^i}} \right)_T^*, \left(\frac{\Sigma^{\Delta\overline{\varphi}^i}}{\Sigma^{\hat{\varphi}^i}} \right)_T^* \right\}, \\
\mathcal{X}_{\mathfrak{E}}(Q) &:= \left\{ (\mathfrak{E}_T^*)^4, \left((\mathfrak{E}^{-1})_T^*\right)^4 \right\}, \\
\mathcal{X}_{\Delta\overline{\varphi}^i}(Q) &:= \Big\{ |\mu^{\hat{\varphi}^i,Q}| \bullet I_T, |\mu^{\Delta\overline{\varphi}^i,Q}| \bullet I_T, \Delta\overline{\varphi}^i |\mu^{\mathfrak{E},Q}| \bullet I_T, ((\Delta\overline{\varphi}^i)^2 \Sigma^{\mathfrak{E}} \bullet I_T)^{1/2}, \\
&\qquad \frac{\Sigma^{\hat{\varphi}^i}}{\Delta\overline{\varphi}^i} \bullet I_T, \frac{\Sigma^{\Delta\overline{\varphi}^i}}{\Delta\overline{\varphi}^i} \bullet I_T, \Sigma^{\hat{\varphi}^i} \bullet I_T, \Sigma^{\Delta\overline{\varphi}^i} \bullet I_T, \Sigma^{\mathfrak{E}} \bullet I_T, \frac{\mathfrak{E}}{\Delta\overline{\varphi}^i} \Sigma^{\hat{\varphi}^i} \bullet I_T \Big\}, \\
\mathcal{X}_{\hat{\varphi}^i}(Q) &:= \left\{ (\hat{\varphi}^i + 1) |\mu^{S,Q}| \bullet I_T, \left(((\hat{\varphi}^i)^2 + 1) \Sigma^S \bullet I_T \right)^{1/2} \right\}.
\end{aligned}
$$

Theorem 3.2 for the agents trading the risky asset is valid under the following assumptions:

Assumption A.2. Fix $i \in \{1,2\}$ and $\ell^i \in \mathbb{R}$. Assume that $\mathcal{X}_{N^i}(\hat{Q}^i) \subset L^1(\hat{Q}^i)$, $\mathcal{X}_{\Delta\overline{\varphi}^i}(\hat{Q}^i)$, $\mathcal{X}_{\mathfrak{E}}(\hat{Q}^i) \subset L^{1+\kappa}(\hat{Q}^i)$ and $\mathcal{X}_{\hat{\varphi}^i}(P) \in L^2(P)$ for some $\kappa > 0$, where $\hat{Q}^i$ is the frictionless dual martingale measure for Agent i. Moreover, suppose that

$$
E^{\hat{Q}^i}\left[\exp\left(2(\gamma^i)^2 \Big(2 + \frac{\kappa}{2} + \frac{2}{\kappa} \Big)^2 \frac{(\hat{\varphi}^i - \ell^i)^2}{A^2} \Sigma^S \bullet I_T \right) \right] < \infty. \tag{A.2}
$$

Finally, assume that $\mu^{\varphi^i,\hat{Q}^i}$, $\mu^{\Delta\overline{\varphi}^i,\hat{Q}^i}$, and $\mu^{\mathfrak{E},\hat{Q}^i}$ are locally bounded.

The exponential moment condition (A.2) is required due to the exponential form of the utilities. For Theorem 3.4 that treats Agent 3 (who receives the transaction costs paid by Agents 1 and 2 and consumes them optimally), we additionally need the following:[28]

Assumption A.3. Assume that $\mathcal{X}_{N^1}(\hat{Q}^3), \mathcal{X}_{N^2}(\hat{Q}^3), \mathcal{X}_{\Delta\overline{\varphi}^1}(\hat{Q}^3), \mathcal{X}_{\Delta\overline{\varphi}^2}(\hat{Q}^3) \subset L^1(\hat{Q}^3)$, where $\hat{Q}^3$ is the frictionless equivalent martingale measure for Agent 3.

A.3 Normalized Strategy Correction and an Ergodic Result

Fix $i \in \{1,2\}$. For $\varepsilon > 0$, define the *normalized strategy correction*

$$
N^{i,\varepsilon} = \frac{\Delta\varphi^{i,\varepsilon}}{\varepsilon^{\frac{1}{3}} \Delta\overline{\varphi}^i}. \tag{A.3}
$$

Note that by definition, $N^{i,\varepsilon}$ takes values in $[-1,1]$ and the positive and negative variations $\varphi^{i,\varepsilon\uparrow}$ and $\varphi^{i,\varepsilon\downarrow}$ of $\varphi^{i,\varepsilon}$ only move on $\{N^{i,\varepsilon} = -1\}$ and $\{N^{i,\varepsilon} = 1\}$, respectively.

For small ε, the reflected Itô process $N^{i,\varepsilon}$ can be approximated by a reflected Brownian motion at the leading order. $N_t^{i,\varepsilon}$ in turn converges weakly to the uniform law on $[-1,1]$ with variance $1/3$ as $\varepsilon \downarrow 0$. This underlies the following ergodic result which is a straightforward generalisation of [1, Lemma 4.3.4]. It plays an important role in the computation of the primal and dual bounds in the proof of Theorem 3.2.

[28] In equilibrium, $\hat{\varphi}^1 + \hat{\varphi}^2 = 1$ and $\Delta\overline{\varphi}^1 = \Delta\overline{\varphi}^2$ imply $\mathcal{X}_{N^1}(\hat{Q}^3) = \mathcal{X}_{N^2}(\hat{Q}^3)$ and $\mathcal{X}_{\Delta\overline{\varphi}^2}(\hat{Q}^3) = \mathcal{X}_{\Delta\overline{\varphi}^1}(\hat{Q}^3)$.

Lemma A.4. *Let $Q \approx P$, and suppose that $\mathcal{X}_{N^i}(Q) \subset L^1(Q)$. Let $p \geq 1$ and $(K_t)_{t\in[0,T]}$ a continuous adapted process satisfying $E^Q\left[(\int_0^T |K_t|\,\mathrm{d}t)^p\right] < \infty$. Then:*

$$\lim_{\varepsilon\downarrow 0}\left\| \int_0^{\cdot} (N_s^{i,\varepsilon})^2 K_s\,\mathrm{d}s - \frac{1}{3}\int_0^{\cdot} K_s\,\mathrm{d}s \right\|_{\mathcal{S}^p(Q)} = 0. \tag{A.4}$$

(The definition of the semimartingale norm $\mathcal{S}^p(Q)$ is recalled in Definition E.1.)

Proof. The difference to [1, Lemma 4.3.4] is that we assume $E^Q[(\int_0^T |K_t|\,\mathrm{d}t)^p] < \infty$ rather than $E^Q[\int_0^T |K_t|\,\mathrm{d}t] < \infty$ to derive convergence in $\mathcal{S}^p(Q)$ instead of convergence in $\mathcal{S}^1(Q)$. [1, Lemma 4.3.4] first establishes convergence of $\sup_{t\in[0,T]}\left|(N^{i,\varepsilon})^2 K \bullet I_t - \frac{1}{3}K \bullet I_t\right|$ to 0 in probability as $\varepsilon \downarrow 0$. It is then shown that $\int_0^T |K_t|\,\mathrm{d}t$ (times a constant) is an almost-sure upper bound. By the dominated convergence theorem, the convergence in probability can thus be lifted to convergence in $L^p(Q)$ for $p \geq 1$ if $E^Q[(\int_0^T |K_t|\,\mathrm{d}t)^p] < \infty$ as is assumed here. □

A.4 Stopping the Normalised Transaction Costs

Before proceeding to the other estimates, we stop the normalized transaction costs $\mathfrak{E}$. This allows to treat them "almost as constant" in many of the estimates below. Moreover, this yields uniform higher-order bound on the transaction costs incurred by liquidating any admissible strategy at any stopping time.

Lemma A.5. *Fix $\kappa > 0$. For $\varepsilon > 0$, define the stopping time*

$$\tau_\varepsilon^{\mathfrak{E}} = \inf\left\{t \in [0,T] : \max\left(\mathfrak{E}_t, \mathfrak{E}_t^{-1}\right) > \varepsilon^{-\frac{1}{6}\left(1-\frac{1}{3}\frac{\kappa}{1+\kappa}\right)}\right\} \wedge T. \tag{A.5}$$

Let $Q \approx P$ and suppose that $\mathcal{X}_{\mathfrak{E}} \subset L^{1+\kappa}(Q)$. Then

$$\lim_{\varepsilon\downarrow 0} \varepsilon^{-\frac{2}{3}(1+\frac{\kappa}{2})} Q[\tau_\varepsilon^{\mathfrak{E}} < T] = 0. \tag{A.6}$$

Proof. Define the stopping times $\tau_\varepsilon^{\mathfrak{E},1} = \inf\left\{t \in [0,T] : \mathfrak{E}_t > \varepsilon^{-\frac{1}{6}\left(1-\frac{1}{3}\frac{\kappa}{1+\kappa}\right)}\right\} \wedge T$ and $\tau_\varepsilon^{\mathfrak{E},2} = \inf\left\{t \in [0,T] : \mathfrak{E}_t^{-1} > \varepsilon^{-\frac{1}{6}\left(1-\frac{1}{3}\frac{\kappa}{1+\kappa}\right)}\right\} \wedge T$. Then $\tau_\varepsilon^{\mathfrak{E}} = \tau_\varepsilon^{\mathfrak{E},1} \wedge \tau_\varepsilon^{\mathfrak{E},2}$ and it suffices to show

$$\lim_{\varepsilon\downarrow 0} \varepsilon^{-\frac{2}{3}(1+\frac{\kappa}{2})} \hat{Q}[\tau_\varepsilon^{\mathfrak{E},j} < T] = 0, \quad \text{for } j \in \{1,2\}. \tag{A.7}$$

For $j = 1$, set $K^\varepsilon := \mathfrak{E}$ and $L := \mathfrak{E}$. Then the claim follows from Lemma E.7(a) with $p = 4(1+\kappa)$, $q = 0$, and $r = -\frac{1}{6}(1-\frac{1}{3}\frac{\kappa}{1+\kappa})$ under the stated integrability assumptions. The case $j = 2$ is proved analogously. □

Lemma A.6. *Let $\hat{Q}$ be an equivalent local martingale measure for S, φ any ε^i-admissible strategy, and $\ell^i \in \mathbb{R}$. Let $(\tau_\varepsilon)_{\varepsilon>0}$ be a family of stopping times with $\tau_\varepsilon \leq \tau_\varepsilon^{\mathfrak{E}}$, for each $\varepsilon > 0$. Then:*

$$E^{\hat{Q}}\left[\sup_{t\in[0,T]} \varepsilon\alpha^i \mathfrak{E}_{t\wedge\tau_\varepsilon}|\varphi_{t\wedge\tau_\varepsilon} - \ell^i|\right] = o(\varepsilon^{\frac{2}{3}}).$$

Proof. The definition of $\tau_\varepsilon^{\mathfrak{E}}$ implies $\sup_{t\in[0,T]}\varepsilon\mathfrak{E}_{t\wedge\tau_\varepsilon}\leq\varepsilon^{1-\frac{1}{6}\left(1-\frac{1}{3}\frac{\kappa}{1+\kappa}\right)}=\varepsilon^{\frac{5}{6}\left(1+\frac{1}{15}\frac{\kappa}{1+\kappa}\right)}$. Thus, it suffices to show $E^{\hat{Q}}\left[\sup_{t\in[0,T]}\varepsilon\mathfrak{E}_{t\wedge\tau_\varepsilon}|\varphi_{t\wedge\tau_\varepsilon}|\right]=o(\varepsilon^{\frac{2}{3}})$. By definition of $\tau_\varepsilon^{\mathfrak{E}}$,

$$\mathfrak{E}_{t\wedge\tau_\varepsilon}\leq\varepsilon^{-\frac{1}{6}\left(1-\frac{1}{3}\frac{\kappa}{1+\kappa}\right)}\leq\varepsilon^{-\frac{1}{6}\left(1-\frac{1}{3}\frac{\kappa}{1+\kappa}\right)}\frac{\varepsilon^{-\frac{1}{6}\left(1-\frac{1}{3}\frac{\kappa}{1+\kappa}\right)}}{\max_{u\in[0,\tau_\varepsilon]}\mathfrak{E}_u^{-1}}=\varepsilon^{-\frac{1}{3}\left(1-\frac{1}{3}\frac{\kappa}{1+\kappa}\right)}\min_{u\in[0,\tau_\varepsilon]}\mathfrak{E}_u.$$

Now using that $E^{\hat{Q}}\left[\min_{u\in[0,\tau_\varepsilon]}\mathfrak{E}_u|\varphi_{u\wedge\tau_\varepsilon}|\right]\leq E^{\hat{Q}}\left[\mathfrak{E}_0\varphi_0+\int_{(0,T]}\mathfrak{E}_t\,\mathrm{d}\|\varphi\|_t\right]<\infty$ by (2.8), we obtain

$$E^{\hat{Q}}\left[\sup_{t\in[0,T]}\varepsilon\mathfrak{E}_{t\wedge\tau_\varepsilon}|\varphi_{t\wedge\tau_\varepsilon}|\right]\leq\varepsilon^{\frac{2}{3}\left(1+\frac{1}{6}\frac{\kappa}{1+\kappa}\right)}E^{\hat{Q}}\left[\mathfrak{E}_0\varphi_0+\int_{(0,T]}\mathfrak{E}_t\,\mathrm{d}\|\varphi\|_t\right]=o(\varepsilon^{\frac{2}{3}}).$$

This completes the proof. □

A.5 Adjustment Process and a First Limit Theorem

For the computation of both the primal lower and dual upper bounds, a crucial role is played by a *shadow price process.* This is a fictitious price process that evolves in the bid-ask spread of the market with transaction costs, but does not lead to more favorable terms of trade because it coincides with the bid or ask price whenever the (asymptotically) optimal strategy prescribes to sell or buy stocks, respectively, compare [16, 47, 40].

In the present context, the deviation of the shadow price from the midprice will be described by the following additive *adjustment process* (this parametrization in terms of $N^{i,\varepsilon}$ from (A.3) is derived heuristically in [41]; it is also used in [39, 1]):

$$\Delta S_t^{i,\varepsilon}:=\frac{1}{2}\varepsilon\alpha^i\mathfrak{E}_t\left((N_t^{i,\varepsilon})^3-3N_t^{i,\varepsilon}\right). \tag{A.8}$$

Itô's formula readily yields the corresponding dynamics:

Proposition A.7. *For each $\varepsilon>0$, $\Delta S^{i,\varepsilon}$ is an Itô process with values in $[-\varepsilon\alpha^i\mathfrak{E},\varepsilon\alpha^i\mathfrak{E}]$ and*

$$\begin{aligned}
\mathrm{d}\Delta S_t^{i,\varepsilon}=&-\frac{3}{2}\varepsilon^{\frac{2}{3}}\frac{\alpha^i\mathfrak{E}_t}{\Delta\overline{\varphi}_t^i}\left((N_t^{i,\varepsilon})^2-1\right)\mathrm{d}\hat{\varphi}_t^i\\
&-\frac{3}{2}\varepsilon\frac{\alpha^i\mathfrak{E}_t}{\Delta\overline{\varphi}_t^i}N_t^{i,\varepsilon}\left((N_t^{i,\varepsilon})^2-1\right)\mathrm{d}\Delta\overline{\varphi}_t^i+\frac{1}{2}\varepsilon\alpha^i\left((N_t^{i,\varepsilon})^3-3N_t^{i,\varepsilon}\right)\mathrm{d}\mathfrak{E}_t\\
&+\frac{3}{2}\varepsilon^{\frac{1}{3}}\frac{\alpha^i\mathfrak{E}_t}{(\Delta\overline{\varphi}_t^i)^2}N_t^{i,\varepsilon}\,\mathrm{d}\langle\hat{\varphi}^i\rangle_t\\
&+\frac{3}{2}\varepsilon^{\frac{2}{3}}\frac{\alpha^i\mathfrak{E}_t}{(\Delta\overline{\varphi}_t^i)^2}\left((N_t^{i,\varepsilon})^2+1\right)\mathrm{d}\langle\hat{\varphi}^i,\Delta\overline{\varphi}^i\rangle_t-\frac{3}{2}\varepsilon^{\frac{2}{3}}\frac{\alpha^i}{\Delta\overline{\varphi}_t^i}\left((N_t^{i,\varepsilon})^2-1\right)\mathrm{d}\langle\hat{\varphi}^i,\mathfrak{E}\rangle_t\\
&+\frac{3}{2}\varepsilon\frac{\alpha^i\mathfrak{E}_t}{(\Delta\overline{\varphi}_t^i)^2}N_t^{i,\varepsilon}\left(2(N_t^{i,\varepsilon})^2-1\right)\mathrm{d}\langle\Delta\overline{\varphi}^i\rangle_t-\frac{3}{2}\varepsilon\alpha^i\frac{N_t^{i,\varepsilon}}{\Delta\overline{\varphi}_t^i}\left((N_t^{i,\varepsilon})^2-1\right)\mathrm{d}\langle\Delta\overline{\varphi}^i,\mathfrak{E}\rangle_t.
\end{aligned}\tag{A.9}$$

The next result uses the adjustment process to compute the (normalised) transaction costs $\int_0^\cdot\varepsilon^{\frac{1}{3}}\alpha^i\mathfrak{E}_s\,\mathrm{d}\|\varphi^{i,\varepsilon}\|_s$ generated by the candidate strategy $\varphi^{i,\varepsilon}$ until the normalized transaction cost $\mathfrak{E}$ is stopped:[29]

[29]Lemma A.8 extends [1, Lemma 4.3.2] in two directions. First, the convergence is in $\mathcal{S}^{1+\kappa}$ rather than $\mathcal{S}^1$, and more importantly, due to the stopping the estimates are often of order $\varepsilon^{\frac{1}{6}}$ lower than in [1, Lemma 4.3.2].

Lemma A.8. *Let $Q \approx P$, $\kappa > 0$ and assume that $\mathcal{X}_{\Delta\overline{\varphi}^i} \subset L^{1+\kappa}(Q)$. Then:*

$$\lim_{\varepsilon\downarrow 0}\left\|\int_0^{\cdot\wedge\tau_\varepsilon^{\mathfrak{E}}} \varepsilon^{\frac{1}{3}}\alpha^i\mathfrak{E}_s\,\mathrm{d}\|\varphi^{i,\varepsilon}\|_s - \frac{3}{2}\int_0^{\cdot\wedge\tau_\varepsilon^{\mathfrak{E}}}\left(1-2(N_s^{i,\varepsilon})^2\right)\frac{\alpha^i\mathfrak{E}_s}{\Delta\overline{\varphi}_s^i}\Sigma_s^{\hat{\varphi}^i}\,\mathrm{d}s\right\|_{\mathcal{S}^{1+\kappa}(Q)} = 0. \tag{A.10}$$

Proof. Recall that $\varphi^{i,\varepsilon\uparrow}$ only moves on $\{N^{i,\varepsilon}=-1\}=\{\Delta S^{i,\varepsilon}=\varepsilon\alpha^i\mathfrak{E}\}$, and $\varphi^{i,\varepsilon\downarrow}$ only moves on $\{N^{i,\varepsilon}=1\}=\{\Delta S^{i,\varepsilon}=-\varepsilon\alpha^i\mathfrak{E}\}$. As a consequence,

$$\begin{aligned}\varepsilon\alpha^i\mathfrak{E}\bullet\|\varphi^{i,\varepsilon}\| &= \varepsilon\alpha^i\mathfrak{E}\bullet\varphi^{i,\varepsilon\uparrow}+\varepsilon\alpha^i\mathfrak{E}\bullet\varphi^{i,\varepsilon\downarrow} = \varepsilon\alpha^i\mathfrak{E}\bullet\varphi^{i,\varepsilon\uparrow}-(-\varepsilon\alpha^i\mathfrak{E})\bullet\varphi^{i,\varepsilon\downarrow}\\ &= \Delta S^{i,\varepsilon}\bullet\varphi^{i,\varepsilon\uparrow}-\Delta S^{i,\varepsilon}\bullet\varphi^{i,\varepsilon\downarrow} = \Delta S^{i,\varepsilon}\bullet\varphi^{i,\varepsilon}.\end{aligned} \tag{A.11}$$

Next, add and subtract $\Delta S^{i,\varepsilon}\bullet\hat{\varphi}^i$ and integrate by parts. Recalling that $\varphi^{i,\varepsilon}-\hat{\varphi}^i=\Delta\varphi^{i,\varepsilon}$, $\Delta\varphi_0^{i,\varepsilon}=0$, and $\varphi^{i,\varepsilon}$ is continuous and of finite variation, this yields

$$\begin{aligned}\Delta S^{i,\varepsilon}\bullet\varphi^{i,\varepsilon} &= \Delta S^{i,\varepsilon}\bullet\hat{\varphi}^i+\Delta S^{i,\varepsilon}\bullet(\varphi^{i,\varepsilon}-\hat{\varphi}^i)\\ &= \Delta S^{i,\varepsilon}\bullet\hat{\varphi}^i+\Delta S^{i,\varepsilon}(\varphi^{i,\varepsilon}-\hat{\varphi}^i)-(\varphi^{i,\varepsilon}-\hat{\varphi}^i)\bullet\Delta S^{i,\varepsilon}-\langle\varphi^{i,\varepsilon}-\hat{\varphi}^i,\Delta S^{i,\varepsilon}\rangle\\ &= \Delta S^{i,\varepsilon}\bullet\hat{\varphi}^i+\Delta S^{i,\varepsilon}\Delta\varphi^{i,\varepsilon}-\Delta\varphi^{i,\varepsilon}\bullet\Delta S^{i,\varepsilon}+\langle\hat{\varphi}^i,\Delta S^{i,\varepsilon}\rangle.\end{aligned} \tag{A.12}$$

We proceed by estimating each summand on the right-hand side of (A.12) on $[\![0,\tau_\varepsilon^{\mathfrak{E}}]\!]$. In doing so, we use Proposition A.7, that $\Delta S^{i,\varepsilon}$ is $[-\varepsilon\alpha^i\mathfrak{E},\varepsilon\alpha^i\mathfrak{E}]$-valued, $\mathfrak{E}\le\varepsilon^{-\frac{1}{6}}$ on $[\![0,\tau_\varepsilon^{\mathfrak{E}}]\!]$, $|\Delta\varphi^{i,\varepsilon}|\le\varepsilon^{\frac{1}{3}}\Delta\overline{\varphi}^i$, $|N^{i,\varepsilon}|\le 1$ and apply Lemma E.4 (with the universal constant C_p). Together with the stated integrability assumptions (using also the inequality $|\Sigma^{X,Y}|\le\frac{1}{2}(\Sigma^X+\Sigma^Y)$ for Itô processes X and Y), this gives

$$\left\|(\Delta S^{i,\varepsilon}\bullet\hat{\varphi}^i)^{\tau_\varepsilon^{\mathfrak{E}}}\right\|_{\mathcal{S}^{1+\kappa}(Q)}\le\varepsilon^{\frac{5}{6}}\alpha^iC_p\left\|\hat{\varphi}^i\right\|_{\mathcal{H}^{1+\kappa}(Q)}=O(\varepsilon^{\frac{5}{6}}), \tag{A.13}$$

$$\left\|\Delta S^{i,\varepsilon}\Delta\varphi^{i,\varepsilon}\mathbf{1}_{[\![0,\tau_\varepsilon^{\mathfrak{E}}]\!]}\right\|_{\mathcal{S}^{1+\kappa}(Q)}\le\varepsilon^{\frac{7}{6}}\alpha^iC_p\left(\left|\Delta\overline{\varphi}_0^i\right|+\left\|\Delta\overline{\varphi}^i\right\|_{\mathcal{H}^{1+\kappa}(Q)}\right)=O(\varepsilon^{7/6}), \tag{A.14}$$

as well as

$$\begin{aligned}&\left\|\left(-\Delta\varphi^{i,\varepsilon}\bullet\Delta S^{i,\varepsilon}+\frac{3}{2}\varepsilon^{\frac{2}{3}}(N^{i,\varepsilon})^2\frac{\alpha^i\mathfrak{E}_s}{\Delta\overline{\varphi}_s^i}\Sigma^{\hat{\varphi}^i}\bullet I\right)^{\tau_\varepsilon^{\mathfrak{E}}}\right\|_{\mathcal{S}^{1+\kappa}(Q)}\le\frac{3}{2}\varepsilon^{\frac{5}{6}}\alpha^iC_p\left\|\hat{\varphi}^i\right\|_{\mathcal{H}^{1+\kappa}(Q)}\\ &\quad+\frac{3}{2}\varepsilon^{\frac{7}{6}}\alpha^iC_p\left\|\Delta\overline{\varphi}^i\right\|_{\mathcal{H}^{1+\kappa}(Q)}+\varepsilon^{4/3}\alpha^iC_p\left\|\Delta\overline{\varphi}^i\bullet\mathfrak{E}\right\|_{\mathcal{H}^{1+\kappa}(Q)}\\ &\quad+3\varepsilon^{\frac{5}{6}}\alpha^i\left\|\frac{1}{\Delta\overline{\varphi}^i}\bullet\langle\hat{\varphi}^i,\Delta\overline{\varphi}^i\rangle\right\|_{\mathcal{S}^{1+\kappa}(Q)}+\frac{3}{2}\varepsilon\alpha^i\left\|\langle\hat{\varphi}^i,\mathfrak{E}\rangle\right\|_{\mathcal{S}^{1+\kappa}(Q)}\\ &\quad+\frac{3}{2}\varepsilon^{\frac{7}{6}}\alpha^i\left\|\frac{1}{\Delta\overline{\varphi}^i}\bullet\langle\Delta\overline{\varphi}^i\rangle\right\|_{\mathcal{S}^{1+\kappa}(Q)}+\frac{3}{2}\varepsilon^{4/3}\alpha^i\left\|\langle\Delta\overline{\varphi}^i,\mathfrak{E}\rangle\right\|_{\mathcal{S}^{1+\kappa}(Q)}=O(\varepsilon^{\frac{5}{6}}),\end{aligned} \tag{A.15}$$

and

$$\begin{aligned}&\left\|\left(\langle\hat{\varphi}^i,\Delta S^{i,\varepsilon}\rangle+\frac{3}{2}\varepsilon^{\frac{2}{3}}\left((N_s^{i,\varepsilon})^2-1\right)\frac{\alpha^i\mathfrak{E}}{\Delta\overline{\varphi}^i}\Sigma^{\hat{\varphi}^i}\bullet I\right)^{\tau_\varepsilon^{\mathfrak{E}}}\right\|_{\mathcal{S}^{1+\kappa}(Q)}\\ &\qquad\le\frac{3}{2}\varepsilon^{\frac{5}{6}}\alpha^i\left\|\frac{1}{\Delta\overline{\varphi}^i}\bullet\langle\hat{\varphi}^i,\Delta\overline{\varphi}^i\rangle\right\|_{\mathcal{S}^{1+\kappa}(Q)}+\varepsilon\alpha^i\left\|\langle\hat{\varphi}^i,\mathfrak{E}\rangle\right\|_{\mathcal{S}^{1+\kappa}(Q)}=O(\varepsilon^{\frac{5}{6}}).\end{aligned} \tag{A.16}$$

The assertion now follows by combining (A.11) – (A.16). $\square$

Together with Lemma A.4 and dominated convergence, Lemma A.8 yields the following formula for the (normalised) transaction costs generated by the candidate strategy $\varphi^{i,\varepsilon}$:

Lemma A.9. *Let $Q \approx P$, $\kappa > 0$, and assume that $\mathcal{X}_{N^i} \subset L^1(Q)$ and $\mathcal{X}_{\Delta\overline{\varphi}^i} \subset L^{1+\kappa}(Q)$. Then:*

$$\lim_{\varepsilon\downarrow 0} \left\| \int_0^{\cdot\wedge\tau_\varepsilon^{\mathfrak{E}}} \varepsilon^{\frac{1}{3}} \alpha^i \mathfrak{E}_t \, \mathrm{d}\|\varphi^{i,\varepsilon}\|_s - \frac{1}{2}\int_0^{\cdot} \frac{\alpha^i \mathfrak{E}_s}{\Delta\overline{\varphi}_s^i} \Sigma_s^{\hat{\varphi}^i} \, \mathrm{d}s \right\|_{\mathcal{S}^{1+\kappa}(Q)} = 0. \tag{A.17}$$

A.6 Primal Stopping

We now introduce auxiliary stopping times at which the candidate strategy $\varphi^{i,\varepsilon}$ is liquidated to a static position $\ell^i \in \mathbb{R}$. This technical modification is needed to control the *direct transaction costs* (directly due to the trading friction) and the *indirect transaction costs* (due to displacement from the frictionless optimizer) by a small but positive power of ε. This will be a crucial element for the proof of the asymptotic expansion in Theorem 3.2. First, we stop the direct transaction costs:

Lemma A.10. *Fix $\kappa > 0$ and $\ell^i \in \mathbb{R}$. For $\varepsilon > 0$, define the stopping time*

$$\tau_\varepsilon^{i,D} := \inf\left\{ t \in [0,T] : \int_0^t \varepsilon \alpha^i \mathfrak{E}_u \, \mathrm{d}\|\varphi^{i,\varepsilon}\|_u + \varepsilon \alpha^i \mathfrak{E}_t |\varphi_t^{i,\varepsilon} - \ell^i| > \alpha^i \varepsilon^{\frac{2}{9}\frac{\kappa}{1+\kappa}} \right\} \wedge \tau_\varepsilon^{\mathfrak{E}}. \tag{A.18}$$

Let $Q \approx P$ and suppose that $\mathcal{X}_{N^i} \subset L^1(Q)$ and $\mathcal{X}_{\Delta\overline{\varphi}^i} \subset L^{1+\kappa}(Q)$. Then:

$$\lim_{\varepsilon\downarrow 0} \varepsilon^{-\frac{2}{3}(1+\frac{\kappa}{2})} Q[\tau_\varepsilon^{i,D} < T] = 0. \tag{A.19}$$

Proof. Define $\tau_\varepsilon^{i,D,1} := \inf\left\{ t \in [0,T] : \int_0^t \varepsilon \alpha^i \mathfrak{E}_u \mathbf{1}_{[\![0,\tau_\varepsilon^{\mathfrak{E}}]\!]} \, \mathrm{d}\|\varphi^{i,\varepsilon}\|_u > \frac{1}{2}\alpha^i \varepsilon^{\frac{2}{9}\frac{\kappa}{1+\kappa}} \right\} \wedge T$ and $\tau_\varepsilon^{i,D,2} := \inf\left\{ t \in [0,T] : \varepsilon \alpha^i \mathfrak{E}_{t\wedge\tau_\varepsilon^{\mathfrak{E}}} |\varphi_{t\wedge\tau_\varepsilon^{\mathfrak{E}}}^{i,\varepsilon} - \ell^i| > \frac{1}{2}\alpha^i \varepsilon^{\frac{2}{9}\frac{\kappa}{1+\kappa}} \right\} \wedge T$. As $\tau_\varepsilon^{i,D} \geq \tau_\varepsilon^{i,D,1} \wedge \tau_\varepsilon^{i,D,2} \wedge \tau^{\mathfrak{E}}$, by (A.7), it suffices to show that, for $j \in \{1,2\}$,

$$\lim_{\varepsilon\downarrow 0} \varepsilon^{-\frac{2}{3}(1+\frac{\kappa}{2})} Q[\tau_\varepsilon^{i,D,j} < T] = 0. \tag{A.20}$$

For $j = 1$, set $K^{\varepsilon,1} := \varepsilon^{\frac{1}{3}} \alpha^i \mathfrak{E} \mathbf{1}_{[\![0,\tau_\varepsilon^{\mathfrak{E}}]\!]} \bullet \|\varphi^{i,\varepsilon}\|$ and $L^1 := \frac{1}{2}\frac{\alpha^i \mathfrak{E}}{\Delta\overline{\varphi}^i} \Sigma^{\hat{\varphi}^i} \bullet I$. Then (A.20) for $j = 1$ follows from Lemma E.7(b) with $p = 1+\kappa$, $q = 2/3$, and $r = \frac{2}{9}\frac{\kappa}{1+\kappa}$ and Lemma A.9 under the stated integrability assumptions.

For $j = 2$, set $K^{\varepsilon,2} := \varepsilon^{\frac{1}{3}} \mathfrak{E}_{\cdot\wedge\tau_\varepsilon^{\mathfrak{E}}} \alpha^i |\varphi_{\cdot\wedge\tau_\varepsilon^{\mathfrak{E}}}^{i,\varepsilon} - \ell^i|$ and $L^2 := \alpha^i(|\hat{\varphi}^i| + |\Delta\overline{\varphi}^i| + |\ell^i|)$. Then (A.20) for $j = 2$ follows from Lemma E.7(a) with $p = 1+\kappa$, $q = \frac{2}{3}$, $r = \frac{2}{9}\frac{\kappa}{1+\kappa}$ and Lemma E.4 under the stated integrability assumptions. $\square$

Next, we stop the indirect transaction costs:

Lemma A.11. *Fix $\kappa > 0$. For $\varepsilon > 0$, define the stopping time*

$$\tau_\varepsilon^{i,I} := \inf\left\{ t \in [0,T] : \left| \int_0^t \Delta\varphi_u^{i,\varepsilon} \, \mathrm{d}S_u \right| > \varepsilon^{\frac{1}{9}\frac{\kappa}{1+\kappa}} \right\} \wedge T. \tag{A.21}$$

Let $\hat{Q}$ be an equivalent local martingale measure for S and $E^{\hat{Q}}\left[\left(\int_0^T \frac{\mathfrak{C}_t}{\Delta\overline{\varphi}^i_t}\Sigma_t^{\hat{\varphi}^i}\,\mathrm{d}t\right)^{1+\kappa}\right] < \infty$. *Then:*

$$\lim_{\varepsilon\downarrow 0} \varepsilon^{-\frac{2}{3}(1+\frac{\kappa}{2})}\hat{Q}[\tau_\varepsilon^{i,I} < T] = 0. \tag{A.22}$$

Proof. Set $K^\varepsilon := \varepsilon^{-1/3}\Delta\varphi^{i,\varepsilon} \bullet S$ and $L := \frac{3}{2}\frac{A}{\gamma^i}\frac{\alpha^i\mathfrak{C}}{\Delta\overline{\varphi}^i}\Sigma^{\hat{\varphi}^i} \bullet I$. Then, using that $|\Delta\varphi^{i,\varepsilon}| \leq \varepsilon^{\frac{1}{3}}\Delta\overline{\varphi}^i$ and the definition (3.6) of $\Delta\overline{\varphi}^i$, we obtain

$$[K^\varepsilon] \leq (\Delta\overline{\varphi}^i)^2\Sigma^S \bullet I = (\Delta\overline{\varphi}^i)^3\frac{\Sigma^S}{\Delta\overline{\varphi}^i} \bullet I = L.$$

Since A is uniformly bounded from above by A_0, the claim now follows from Lemma E.7(c) with $p = 2(1+\kappa)$, $q = 1/3$, and $r = \frac{1}{9}\frac{\kappa}{1+\kappa}$ under the stated integrability assumption. ☐

We can now define the "primal stopping time" used in the definition of the candidate optimal consumption pair in Theorem 3.2.

Definition A.12. Let $\ell^i \in \mathbb{R}$ and $\kappa > 0$. Then the *primal stopping time* is defined as

$$\hat{\tau}_\varepsilon^i := \tau_\varepsilon^{i,D} \wedge \tau_\varepsilon^{i,I} > 0, \tag{A.23}$$

where $\tau_\varepsilon^{i,D}$ is defined in (A.18) and $\tau_\varepsilon^{i,I}$ is defined in (A.21).

A.7 Dual Stopping

To obtain a dual upper bound, we now construct a shadow price $\hat{S}^{i,\varepsilon}$ and a corresponding dual martingale $\overline{Z}^{i,\varepsilon}$ (the density process of the corresponding dual minimizer relative to the frictionless marginal pricing measure $\hat{Q}^i$). A candidate for the shadow price is the process

$$S^{i,\varepsilon} := S + \Delta S^{i,\varepsilon},$$

which takes values in the bid-ask spread by construction. Similarly as in [39], to derive a candidate for the dual martingale we proceed as follows: we first write

$$S^{i,\varepsilon} = \mu^{\Delta S^{i,\varepsilon},\hat{Q}^i} \bullet I + \sigma^{\Delta S^{i,\varepsilon}} \bullet W^{\delta,\hat{Q}^i} + S^{i,\varepsilon\perp},$$

where $\mu^{\Delta S^{i,\varepsilon},\hat{Q}^i}$ denotes the $\hat{Q}^i$-drift of $S^{i,\varepsilon}$, $\sigma^{\Delta S^{i,\varepsilon}}$ denotes the "S-hedgeable" volatility component relative to the $\hat{Q}^i$-Brownian motion $W^{\delta,\hat{Q}^i}$ driving S under $\hat{Q}^i$, and $S^{i,\varepsilon\perp}$ is a local $\hat{Q}^i$-martingale orthogonal to $W^{\delta,\hat{Q}^i}$. Note that $\sigma^{\Delta S^{i,\varepsilon}} = \frac{\Sigma^S + \Sigma^{\Delta S^{i,\varepsilon},S}}{\sqrt{\Sigma^S}}$ and write

$$\lambda^{i,\varepsilon} := \frac{\mu^{\Delta S^{i,\varepsilon},\hat{Q}^i}}{\sigma^{\Delta S^{i,\varepsilon}}} = \frac{\mu^{\Delta S^{i,\varepsilon},\hat{Q}^i}\sqrt{\Sigma^S}}{\Sigma^S + \Sigma^{\Delta S^{i,\varepsilon},S}}$$

for the *adjustment market price of risk* (under $\hat{Q}^i$). In view of Girsanov's theorem, we would then like to use

$$\overline{Z}^{i,\varepsilon} := \left(-\lambda^{i,\varepsilon} \bullet W^{\delta,\hat{Q}^i}\right).$$

as a dual martingale density. However, in order to guarantee that $\lambda^{i,\varepsilon}$ is well defined (the denominator could become zero) and to ensure enough integrability of $\overline{Z}^{i,\varepsilon}$ for our asymptotic analysis, some additional stopping is necessary. We first control the adjustment market price of risk:

Lemma A.13. *For $\varepsilon > 0$, define the stopping time*

$$\tau_\varepsilon^{i,\lambda} = \tau_\varepsilon^{i,\lambda,1} \wedge \tau_\varepsilon^{i,\lambda,2} \wedge \tau_\varepsilon^{i,\lambda,3} \wedge \tau_\varepsilon^{i,\lambda,4},$$

where

$$\tau_\varepsilon^{i,\lambda,1} = \inf\left\{ t \in [0,T] : \left| \frac{\Sigma_t^{\Delta S^{i,\varepsilon},S}}{\Sigma_t^S} \right| > \varepsilon^{\frac{1}{3}} \right\} \wedge T, \tag{A.24}$$

$$\tau_\varepsilon^{i,\lambda,2} = \inf\left\{ t \in [0,T] : \left| \frac{3}{2}\varepsilon^{\frac{1}{3}} \frac{\alpha^i \mathfrak{E}_t}{(\Delta\overline{\varphi}_t^i)^2} N_t^{i,\varepsilon} \frac{\Sigma_t^{\hat{\varphi}^i}}{\sqrt{\Sigma_t^S}} \right| > \varepsilon^{\frac{1}{6}} \right\} \wedge T, \tag{A.25}$$

$$\tau_\varepsilon^{i,\lambda,3} = \inf\left\{ t \in [0,T] : \left| \frac{\mu_t^{\Delta S^{i,\varepsilon},\hat{Q}^i}}{\sqrt{\Sigma_t^S}} - \frac{3}{2}\varepsilon^{\frac{1}{3}} \frac{\alpha^i \mathfrak{E}_t}{(\Delta\overline{\varphi}_t^i)^2} N_t^{i,\varepsilon} \frac{\Sigma_t^{\hat{\varphi}^i}}{\sqrt{\Sigma_t^S}} \right| > \varepsilon^{\frac{1}{3}} \right\} \wedge T, \tag{A.26}$$

$$\tau_\varepsilon^{i,\lambda,4} = \inf\left\{ t \in [0,T] : \left| \frac{\left(\mu_t^{\Delta S^{i,\varepsilon},\hat{Q}^i}\right)^2}{\Sigma_t^S} \frac{A}{\gamma^i} - \frac{3}{2}\varepsilon^{\frac{2}{3}} \frac{\alpha^i \mathfrak{E}}{\Delta\overline{\varphi}^i} \Sigma^{\hat{\varphi}^i} (N^{i,\varepsilon})^2 \right| > \varepsilon^{\frac{5}{6}} \right\} \wedge T. \tag{A.27}$$

Assume that $\mu^{\varphi^i,\hat{Q}^i}$, $\mu^{\Delta\overline{\varphi}^i,\hat{Q}^i}$, and $\mu^{\mathfrak{E},\hat{Q}^i}$ are locally bounded. Then:

$$\lim_{\varepsilon \downarrow 0} \hat{Q}^i[\tau_\varepsilon^{i,\lambda} < T] = 0. \tag{A.28}$$

Proof. It suffices to show that for $j \in \{1, \ldots, 4\}$,

$$\lim_{\varepsilon \downarrow 0} \hat{Q}^i[\tau_\varepsilon^{i,\lambda,j} < T] = 0. \tag{A.29}$$

To this end, set

$$\begin{aligned}
K^{\varepsilon,1} &:= \varepsilon^{-\frac{2}{3}} \left(\frac{\Sigma_t^{\Delta S^{i,\varepsilon},S}}{\Sigma_t^S} \right), \\
K^{\varepsilon,2} &:= \varepsilon^{-\frac{1}{3}} \left(\frac{3}{2}\varepsilon^{\frac{1}{3}} \frac{\alpha^i \mathfrak{E}}{(\Delta\overline{\varphi}^i)^2} N^{i,\varepsilon} \frac{\Sigma^{\hat{\varphi}^i}}{\sqrt{\Sigma^S}} \right), \\
K^{\varepsilon,3} &= \varepsilon^{-\frac{2}{3}} \left(\frac{\mu_t^{\Delta S^{i,\varepsilon},\hat{Q}^i}}{\sqrt{\Sigma_t^S}} - \frac{3}{2}\varepsilon^{\frac{1}{3}} \frac{\alpha^i \mathfrak{E}_t}{(\Delta\overline{\varphi}_t^i)^2} N_t^{i,\varepsilon} \frac{\Sigma_t^{\hat{\varphi}^i}}{\sqrt{\Sigma_t^S}} \right), \\
K^{\varepsilon,4} &= \varepsilon^{-1} \left(\frac{\left(\mu_t^{\Delta S^{i,\varepsilon},\hat{Q}^i}\right)^2}{\Sigma_t^S} \frac{A}{\gamma^i} - \frac{3}{2}\varepsilon^{\frac{2}{3}} \frac{\alpha^i \mathfrak{E}}{\Delta\overline{\varphi}^i} \Sigma^{\hat{\varphi}^i} (N^{i,\varepsilon})^2 \right).
\end{aligned}$$

Recalling the dynamics (A.9) of $\Delta S^{i,\varepsilon}$ and the bound $|N^{i,\varepsilon}| \leq 1$ gives for $\varepsilon \in (0,1)$,

$$
\begin{aligned}
|K^{\varepsilon,1}| &\leq \frac{3}{2}\frac{\alpha^i \mathfrak{E}_t}{\Delta\overline{\varphi}^i}\frac{\Sigma^{\hat{\varphi}^i,S}}{\Sigma^S} + \frac{3}{2}\frac{\alpha^i \mathfrak{E}}{\Delta\overline{\varphi}^i}\frac{\Sigma^{\Delta\overline{\varphi}^i,S}}{\Sigma^S} + \alpha^i \frac{\Sigma^{\mathfrak{E},S}}{\Sigma^S} =: L^1, \\
|K^{\varepsilon,2}| &\leq \frac{3}{2}\frac{\alpha^i \mathfrak{E}}{(\Delta\overline{\varphi}^i)^2}\frac{\Sigma^{\hat{\varphi}^i}}{\sqrt{\Sigma^S}} := L^2, \\
|K^{\varepsilon,3}| &\leq \frac{3}{2}\frac{\alpha^i \mathfrak{E}}{\Delta\overline{\varphi}^i}\frac{|\mu^{\hat{\varphi}^i,\hat{Q}^i}|}{\sqrt{\Sigma^S}} + \frac{3}{2}\frac{\alpha^i \mathfrak{E}}{\Delta\overline{\varphi}^i}\frac{|\mu^{\Delta\overline{\varphi}^i,\hat{Q}^i}|}{\sqrt{\Sigma^S}} + \alpha^i \frac{|\mu^{\mathfrak{E},\hat{Q}^i}|}{\sqrt{\Sigma^S}} + 3\frac{\alpha^i \mathfrak{E}}{(\Delta\overline{\varphi}^i)^2}\frac{|\Sigma^{\hat{\varphi}^i,\Delta\overline{\varphi}^i}|}{\sqrt{\Sigma^S}} \\
&\quad + \frac{3}{2}\frac{\alpha^i}{\Delta\overline{\varphi}^i}\frac{|\Sigma^{\hat{\varphi}^i,\mathfrak{E}}|}{\sqrt{\Sigma^S}} + \frac{3}{2}\frac{\alpha^i \mathfrak{E}}{(\Delta\overline{\varphi}^i)^2}\frac{\Sigma^{\Delta\overline{\varphi}^i}}{\sqrt{\Sigma^S}} + \frac{3}{2}\frac{\alpha^i}{\Delta\overline{\varphi}^i}\frac{|\Sigma^{\Delta\overline{\varphi}^i,\mathfrak{E}}|}{\sqrt{\Sigma^S}} =: L^3
\end{aligned}
$$

Now, note hat

$$
K^{\varepsilon,3} = \varepsilon^{-1/3}\left(\varepsilon^{-1/3}\frac{\mu^{\Delta S^{i,\varepsilon},\hat{Q}^i}}{\sqrt{\Sigma^S}} - K^{\varepsilon,2}\right), \quad (K^{\varepsilon,2})^2 = \frac{3}{2}\frac{\gamma^i}{A}(N^{i,\varepsilon})^2 \frac{\alpha^i \mathfrak{E}}{\Delta\overline{\varphi}^i}\Sigma^{\hat{\varphi}^i},
$$

and use the identity $a^2 - b^2 = (a-b)(a-b+2b)$, obtaining

$$
\begin{aligned}
|K^{\varepsilon,4}| &= \varepsilon^{-\frac{1}{3}}\frac{A}{\gamma^i}\left|\left(\varepsilon^{-\frac{2}{3}}\left(\frac{\mu^{\Delta S^{i,\varepsilon}}}{\sqrt{\Sigma^S}}\right)^2 - (K^{\varepsilon,2})^2\right)\right| \\
&= \varepsilon^{-\frac{1}{3}}\frac{A}{\gamma^i}\left|\varepsilon^{\frac{1}{3}}K^{\varepsilon,3}\left(\varepsilon^{\frac{1}{3}}K^{\varepsilon,3} + 2K^{\varepsilon,2}\right)\right| \\
&\leq \frac{A}{\gamma^i}|K^{\varepsilon,3}|\left(|K^{\varepsilon,3}| + 2|K^{\varepsilon,2}|\right) \\
&\leq \frac{A}{\gamma^i}L^3(L^3 + 2L^2) := L^4,
\end{aligned}
$$

for $\varepsilon \in (0,1)$. As $L^1, \dots, L^4$ do not depend on ε and are locally bounded, this together with the definitions of $\tau_\varepsilon^{i,\lambda,j}$ gives (A.29) for $j \in \{1, \dots, 4\}$. □

With the notation from Lemma A.13, define the (*stopped*) adjustment market price of risk by

$$
\lambda^{i,\varepsilon} = \frac{\mu^{\Delta S^{i,\varepsilon},\hat{Q}^i}\sqrt{\Sigma^S}}{\Sigma^S + \Sigma^{\Delta S^{i,\varepsilon},S}}\mathbf{1}_{[\![0,\tau_\varepsilon^{i,\lambda}]\!]}, \quad \varepsilon > 0. \tag{A.30}
$$

The integrability ensured by stopping allows to establish the following ergodic theorem, which will be used in the dual considerations of the proof of Theorem 3.2.

Lemma A.14. *Fix $\kappa > 0$ and suppose $\mathcal{X}_{N^i}(\hat{Q}^i) \subset L^1(\hat{Q}^i)$ and$E^{\hat{Q}^i}\left[\left(\int_0^T \frac{\mathfrak{E}_t}{\Delta\overline{\varphi}_t^i}\Sigma_t^{\hat{\varphi}^i}\,\mathrm{d}t\right)^{1+\kappa}\right] < \infty$. Then:*

$$
\lim_{\varepsilon\downarrow 0}\left\|\int_0^\cdot \varepsilon^{-\frac{2}{3}}(\lambda_s^{i,\varepsilon})^2\frac{A_s}{\gamma^i}\,\mathrm{d}s - \frac{1}{2}\int_0^\cdot \frac{\alpha^i \mathfrak{E}_s}{\Delta\overline{\varphi}_s^i}\Sigma_s^{\hat{\varphi}^i}\,\mathrm{d}s\right\|_{\mathcal{S}^{1+\kappa}(\hat{Q}^i)} = 0. \tag{A.31}
$$

Proof. Set $\tilde{\lambda}^{i,\varepsilon} = \frac{\mu^{\Delta S^{i,\varepsilon},\hat{Q}^i}}{\sqrt{\Sigma^S}}\mathbf{1}_{[\![0,\tau_\varepsilon^{i,\lambda}]\!]}$. It suffices to show (A.31) with $\lambda^{i,\varepsilon}$ replaced by $\tilde{\lambda}^{i,\varepsilon}$. Indeed, (A.31) with $\lambda^{i,\varepsilon}$ replaced by $\tilde{\lambda}^{i,\varepsilon}$ implies that

$$
\lim_{\varepsilon\downarrow 0}\left\|\int_0^\cdot \varepsilon^{-\frac{2}{3}}(\tilde{\lambda}_s^{i,\varepsilon})^2\frac{A_s}{\gamma^i}\,\mathrm{d}s\right\|_{\mathcal{S}^{1+\kappa}(\hat{Q}^i)} = \left\|\frac{1}{2}\int_0^\cdot \frac{\alpha^i \mathfrak{E}_s}{\Delta\overline{\varphi}_s^i}\Sigma_s^{\hat{\varphi}^i}\,\mathrm{d}s\right\|_{\mathcal{S}^{1+\kappa}(\hat{Q}^i)} < \infty.
$$

Moreover, by the definition of $\tau_\varepsilon^{i,\lambda,1}$,

$$|\tilde{\lambda}^{i,\varepsilon}| = |\lambda^{i,\varepsilon}| \left| \frac{\Sigma^S + \Sigma^{\Delta S^{i,\varepsilon},S}}{\Sigma^S} \right| \in |\lambda^{i,\varepsilon}| \left(1 - \varepsilon^{\frac{1}{3}}, 1 + \varepsilon^{\frac{1}{3}}\right).$$

Thus, for $\varepsilon \in (0, \frac{1}{8})$, we have $\left|\left(\tilde{\lambda}^{i,\varepsilon}\right)^2 - \left(\lambda^{i,\varepsilon}\right)^2\right| \leq 3\varepsilon^{\frac{1}{3}} \left(\lambda^{i,\varepsilon}\right)^2 \leq 6\varepsilon^{\frac{1}{3}} \left(\tilde{\lambda}^{i,\varepsilon}\right)^2$, and in turn

$$\begin{aligned}
&\limsup_{\varepsilon \downarrow 0} \left\| \int_0^{\cdot} \varepsilon^{-\frac{2}{3}} (\tilde{\lambda}_s^{i,\varepsilon})^2 \frac{A_s}{\gamma^i} \, \mathrm{d}s - \int_0^{\cdot} \varepsilon^{-\frac{2}{3}} (\lambda_s^{i,\varepsilon})^2 \frac{A_s}{\gamma^i} \, \mathrm{d}s \right\|_{\mathcal{S}^{1+\kappa}(\hat{Q}^i)} \\
&\quad \leq \lim_{\varepsilon \downarrow 0} 6\varepsilon^{\frac{1}{3}} \left\| \int_0^{\cdot} \varepsilon^{-\frac{2}{3}} (\tilde{\lambda}_s^{i,\varepsilon})^2 \frac{A_s}{\gamma^i} \, \mathrm{d}s \right\|_{\mathcal{S}^{1+\kappa}(\hat{Q}^i)} = 0.
\end{aligned}$$

Next, it follows from dominated convergence (using Lemma A.13) and Lemma A.4 that

$$\lim_{\varepsilon \downarrow 0} \left\| \frac{3}{2} \int_0^{\cdot} \frac{\alpha^i \mathfrak{E}_s}{\Delta \overline{\varphi}_s^i} \Sigma_s^{\hat{\varphi}^i} (N_s^{i,\varepsilon})^2 \mathbf{1}_{[\![0,\tau_\varepsilon^{i,\lambda}]\!]} \, \mathrm{d}s - \frac{1}{2} \int_0^{\cdot} \frac{\alpha^i \mathfrak{E}_s}{\Delta \overline{\varphi}_s^i} \Sigma_s^{\hat{\varphi}^i} \, \mathrm{d}s \right\|_{\mathcal{S}^{1+\kappa}(\hat{Q}^i)}.$$

Thus, it suffices to show that

$$\lim_{\varepsilon \downarrow 0} \left\| \int_0^{\cdot} \varepsilon^{-\frac{2}{3}} (\tilde{\lambda}_s^{i,\varepsilon})^2 \frac{A_s}{\gamma^i} \, \mathrm{d}s - \frac{3}{2} \int_0^{\cdot} \frac{\alpha^i \mathfrak{E}_s}{\Delta \overline{\varphi}_s^i} \Sigma_s^{\hat{\varphi}^i} (N_s^{i,\varepsilon})^2 \mathbf{1}_{[\![0,\tau_\varepsilon^{i,\lambda}]\!]} \, \mathrm{d}s \right\|_{\mathcal{S}^{1+\kappa}(\hat{Q}^i)} = 0. \tag{A.32}$$

By definition of $\tau_\varepsilon^{i,\lambda,4}$, it follows that

$$\left| \varepsilon^{-\frac{2}{3}} (\tilde{\lambda}^{i,\varepsilon})^2 \frac{A}{\gamma^i} - \frac{3}{2} \frac{\alpha^i \mathfrak{E}}{\Delta \overline{\varphi}^i} \Sigma^{\hat{\varphi}^i} (N^{i,\varepsilon})^2 \mathbf{1}_{[\![0,\tau_\varepsilon^{i,\lambda}]\!]} \right| \leq \varepsilon^{\frac{1}{6}}.$$

Now (A.32) follows from dominated convergence. □

By a further stopping argument, we can ensure that the stochastic exponential of the (stopped) adjustment market price of risk is uniformly close to 1. This is exploited in Lemmas A.17 and A.18 below.

Lemma A.15. *For $\varepsilon > 0$, define the stopping time*

$$\tau_\varepsilon^{i,\Delta Z} := \inf \left\{ t \in [0,T] : \left| \mathcal{E}\left(\lambda^{i,\varepsilon} \bullet W^{\delta,\hat{Q}^i}\right) - 1 \right| > \frac{1}{2} \varepsilon^{\frac{1}{12}} \right\} \wedge T.$$

Then:

$$\lim_{\varepsilon \downarrow 0} \hat{Q}^i[\tau_\varepsilon^{i,\Delta Z} < T] = 0. \tag{A.33}$$

Proof. Define the stopping times $\tau_\varepsilon^{i,\Delta Z,1} := \inf \left\{ t \in [0,T] : \left| \lambda^{i,\varepsilon} \bullet W_t^{\delta,\hat{Q}^i} \right| > \frac{1}{8} \varepsilon^{\frac{1}{12}} \right\} \wedge T$ and $\tau_\varepsilon^{i,\Delta Z,2} := \inf \left\{ t \in [0,T] : \int_0^t (\lambda_u^{i,\varepsilon})^2 \, \mathrm{d}u > \frac{1}{4} \varepsilon^{\frac{1}{6}} \right\} \wedge T$. Then by definition of the stochastic exponential and the inequality $|\exp(x) - 1| \leq 2|x|$ for $x \in (-\infty, 1)$, we have $\tau_\varepsilon^{i,\Delta Z} \leq \tau_\varepsilon^{i,\Delta Z,1} \wedge \tau_\varepsilon^{i,\Delta Z,2}$ for $\varepsilon \in (0,1)$. Thus, it suffices to show for $j \in \{1,2\}$ that

$$\lim_{\varepsilon \downarrow 0} \hat{Q}^i[\tau_\varepsilon^{i,\Delta Z,j} < T] = 0. \tag{A.34}$$

To this end, note that by definition of $\tau_\varepsilon^{i,\lambda,1}$, $\tau_\varepsilon^{i,\lambda,2}$, and $\tau_\varepsilon^{i,\lambda,3}$, for $\varepsilon \in (0,\frac{1}{8})$,

$$
\begin{aligned}
|\lambda^{i,\varepsilon}| &= \left|\frac{\mu^{\Delta S^{i,\varepsilon},\hat{Q}^i}}{\sqrt{\Sigma^S}}\right| \Big/ \left|1 + \frac{\Sigma^{\Delta S^{i,\varepsilon},S}}{\Sigma^S}\right| \mathbf{1}_{[\![0,\tau_\varepsilon^{i,\lambda}]\!]} \leq \frac{1}{1-\varepsilon^{\frac{1}{3}}} \left|\frac{\mu^{\Delta S^{i,\varepsilon},\hat{Q}^i}}{\sqrt{\Sigma^S}}\right| \mathbf{1}_{[\![0,\tau_\varepsilon^{i,\lambda}]\!]} \\
&\leq 2\left(\left|\frac{\mu^{\Delta S^{i,\varepsilon},\hat{Q}^i}}{\sqrt{\Sigma^S}} - \frac{3}{2}\varepsilon^{\frac{1}{3}}\frac{\alpha^i\mathfrak{E}}{(\Delta\overline{\varphi}^i)^2}N^{i,\varepsilon}\frac{\Sigma^{\hat{\varphi}^i}}{\sqrt{\Sigma^S}}\right| + \left|\frac{3}{2}\varepsilon^{\frac{1}{3}}\frac{\alpha^i\mathfrak{E}}{(\Delta\overline{\varphi}^i)^2}N^{i,\varepsilon}\frac{\Sigma^{\hat{\varphi}^i}}{\sqrt{\Sigma^S}}\right|\right)\mathbf{1}_{[\![0,\tau_\varepsilon^{i,\lambda}]\!]} \\
&\leq 2(\varepsilon^{\frac{1}{3}} + \varepsilon^{\frac{1}{6}}) \leq 4\varepsilon^{\frac{1}{6}}.
\end{aligned} \tag{A.35}
$$

Set $K^{\varepsilon,1} := \varepsilon^{-\frac{1}{6}}(\lambda^{i,\varepsilon} \bullet W^{\delta,\hat{Q}^i})$ and $K^{\varepsilon,2} := \varepsilon^{-\frac{1}{3}}((\lambda^{i,\varepsilon})^2 \bullet I)$, and define $L^1 = L^2 := 16I$. Then by (A.35), $[K^{\varepsilon,1}] \leq L^1$ and $|K^{2,\varepsilon}| \leq L^2$ for $\varepsilon \in (0,\frac{1}{8})$. As $(L^1)^*_T = (L^2)^*_T = 16T$ has moments of all orders, (A.34) for $j = 1,2$ follows from Lemma E.7(a) and (b). $\square$

We can now define the "dual stopping time" used in the definition of the shadow price $\hat{S}^{i,\varepsilon}$ and the dual martingale $\overline{Z}^{i,\varepsilon}$ below.

Definition A.16. Let $\kappa > 0$. Then the "dual stopping time" is defined as

$$
\tilde{\tau}_\varepsilon^i := \tau_\varepsilon^{i,\lambda} \wedge \tau_\varepsilon^{i,\Delta Z} \wedge \tau_\varepsilon^{\mathfrak{E}}, \quad \varepsilon > 0. \tag{A.36}
$$

A.8 Shadow Price and Dual Martingale

For $\varepsilon > 0$, define

$$
\begin{aligned}
\Delta\hat{S}^{i,\varepsilon} &:= (\Delta S^{i,\varepsilon})^{\tilde{\tau}_\varepsilon^i}, \\
\hat{S}^{i,\varepsilon} &:= S + \Delta\hat{S}^{i,\varepsilon}, \\
\overline{Z}^{i,\varepsilon} &= \mathcal{E}\left(-\lambda^{i,\varepsilon}\mathbf{1}_{[\![0,\tilde{\tau}_\varepsilon^i]\!]} \bullet W^{\delta,\hat{Q}^i}\right), \\
\Delta\hat{Z}^{i,\varepsilon} &:= \overline{Z}^{i,\varepsilon} - 1.
\end{aligned}
$$

By definition of $\tau_\varepsilon^{i,\Delta Z}$ and $\tau_\varepsilon^{i,\Delta Z} \leq \tilde{\tau}_\varepsilon^i$, $\overline{Z}^{i,\varepsilon}$ and $\Delta\hat{Z}^{i,\varepsilon}$ are bounded $\hat{Q}$-martingales and

$$
|\Delta\hat{Z}^{i,\varepsilon}| \leq \frac{1}{2}\varepsilon^{\frac{1}{12}}. \tag{A.37}
$$

Girsanov's theorem shows that $\hat{S}^{i,\varepsilon}$ is a local $\hat{Q}^{i,\varepsilon}$-martingale under the measure with density $\mathrm{d}\hat{Q}^{i,\varepsilon} = \overline{Z}_T^{i,\varepsilon}\,\mathrm{d}\hat{Q}^i$. This measure will serve as the dual element leading to an upper bound for the asymptotic expansion in the proof of Theorem 3.2

The "shadow price" $\hat{S}^{i,\varepsilon}$ is guaranteed to take values in the bid-ask spread only until the dual stopping time $\tilde{\tau}_\varepsilon^i$.[30] Nevertheless, it can be used to bound frictional gains processes by their frictionless counterparts – up to a liquidation term of higher order:

Lemma A.17. *Let φ be any ε^i-admissible strategy . Then $\varphi \in L(\hat{S}^{i,\varepsilon})$ and*

$$
\int_0^t \varphi_u\,\mathrm{d}S_u - \int_0^t \varepsilon\alpha^i\mathfrak{E}_u\,\mathrm{d}\|\varphi\|_u \leq \int_0^t \varphi_u\,\mathrm{d}\hat{S}_u^{i,\varepsilon} + \alpha^i\varepsilon\mathfrak{E}_{\tilde{\tau}_\varepsilon\wedge t}|\varphi_{\tilde{\tau}_\varepsilon\wedge t}|, \quad t \in [0,T]. \tag{A.38}
$$

[30] Note that the shadow price does not depend on the primal stopping time $\hat{\tau}_\varepsilon^i$. This is different from the analysis in [1, 39].

Moreover, $\varphi \bullet \hat{S}^{i,\varepsilon}$ is a $\hat{Q}^{i,\varepsilon}$-supermartingale and

$$E^{\hat{Q}^{i,\varepsilon}}\left[\int_0^T \varphi_u \,\mathrm{d}S_u - \int_0^T \varepsilon\alpha^i \mathfrak{E}_u \,\mathrm{d}\|\varphi\|_u\right] \leq o(\varepsilon^{\frac{2}{3}}). \tag{A.39}$$

Proof. As any ε^i-admissible strategy is of finite variation, it is integrable with respect to any semimartingale, and so in particular with respect to $\hat{S}^{i,\varepsilon}$.

Fix $t \in [0,T]$. An integration by parts,[31] $\Delta S_0^{i,\varepsilon} = 0$ and $\Delta S^{i,\varepsilon} \in [-\varepsilon\alpha^i\mathfrak{E}, \varepsilon\alpha^i\mathfrak{E}]$ yield

$$\begin{aligned}\int_0^t \varphi_u \,\mathrm{d}\Delta\hat{S}_u^{i,\varepsilon} &= \int_0^{t\wedge\tilde{\tau}_\varepsilon} \varphi_u \,\mathrm{d}\Delta S_u^{i,\varepsilon} = \varphi_{t\wedge\tilde{\tau}_\varepsilon^i}\Delta S_{t\wedge\tilde{\tau}_\varepsilon^i}^{i,\varepsilon} - \varphi_0\Delta S_0^{i,\varepsilon} - \int_0^{t\wedge\tilde{\tau}^i\varepsilon} \Delta S_u^{i,\varepsilon}\,\mathrm{d}\varphi_u \\ &\geq -\varepsilon\alpha^i\mathfrak{E}_{\tilde{\tau}_\varepsilon^i\wedge t}|\varphi_{\tilde{\tau}_\varepsilon^i\wedge t}| - \int_0^t \varepsilon\alpha^i\mathfrak{E}_u \,\mathrm{d}\|\varphi\|_u.\end{aligned} \tag{A.40}$$

Adding $\int_0^t \varphi_u \,\mathrm{d}S_u$ and rearranging therefore gives (A.38).

To establish that $\varphi \bullet \hat{S}^{i,\varepsilon}$ is a $\hat{Q}^{i,\varepsilon}$-supermartingale, by [25, Proposition 2.2] it suffices to show that $(\varphi \bullet S^{i,\varepsilon})^- = \max(-\varphi \bullet S^{i,\varepsilon}, 0)$ is of class D under $\hat{Q}^{i,\varepsilon}$. As $\frac{\mathrm{d}\hat{Q}^{i,\varepsilon}}{\mathrm{d}\hat{Q}^i} = \overline{Z}_T^{i,\varepsilon}$ is bounded, it suffices to show that it is of class D under $\hat{Q}^i$. (A.38) implies that

$$(\varphi \bullet S^{i,\varepsilon})^- \leq \varepsilon\alpha^i\mathfrak{E}_{\tilde{\tau}_\varepsilon\wedge t}|\varphi_{\tilde{\tau}_\varepsilon\wedge t}| + (\varphi \bullet S - \varepsilon\alpha^i\mathfrak{E} \bullet \|\varphi\|)^-. \tag{A.41}$$

By Lemma A.6, the first term on the right-hand side of (A.41) is of class D. The second term is also of class D because $\varphi \bullet S - \varepsilon\alpha^i\mathfrak{E} \bullet \|\varphi\|$ is a $\hat{Q}^i$-supermartingale (this uses that φ is ε^i-admissible). As processes of class D form a vector space, this shows that $\varphi \bullet \hat{S}^{i,\varepsilon}$ is a $\hat{Q}^{i,\varepsilon}$-supermartingale. Together with (A.38) and Lemma A.6, this gives (A.39). □

We next compute the expectation of the frictionless gains $\hat{\varphi}^i \bullet S_T$ under the dual martingale measure $\hat{Q}^{i,\varepsilon}$:

Lemma A.18. *Assume that $\mathcal{X}_{N^i} \subset L^1(\hat{Q}^i)$ and $\mathcal{X}_{\Delta\overline{\varphi}^i} \subset L^1(\hat{Q}^i)$. Then:*

$$E^{\hat{Q}^{i,\varepsilon}}\left[\hat{\varphi}^i \bullet S_T\right] = \varepsilon^{\frac{2}{3}}\hat{Z}_0^i E^{\hat{Q}}\left[\int_0^T \frac{\alpha^i\mathfrak{E}_s}{\Delta\overline{\varphi}_s^i}\Sigma_s^{\hat{\varphi}^i}\,\mathrm{d}s\right] + o(\varepsilon^{\frac{2}{3}}).$$

Proof. As $S = \hat{S}^{i,\varepsilon} - \Delta\hat{S}^{i,\varepsilon}$, it suffices to show that

$$E^{\hat{Q}^{i,\varepsilon}}\left[\hat{\varphi}^i \bullet \hat{S}_T^{i,\varepsilon}\right] = 0 \tag{A.42}$$

$$E^{\hat{Q}^{i,\varepsilon}}\left[\hat{\varphi}^i \bullet \Delta\hat{S}_T^{i,\varepsilon}\right] = -\varepsilon^{\frac{2}{3}}\hat{Z}_0^i E^{\hat{Q}^i}\left[\int_0^T \frac{\alpha^i\mathfrak{E}_s}{\Delta\overline{\varphi}_s^i}\Sigma_s^{\hat{\varphi}^i}\,\mathrm{d}s\right] + o(\varepsilon^{\frac{2}{3}}). \tag{A.43}$$

In order to establish (A.42), it suffices to show that the local $\hat{Q}^{i,\varepsilon}$-martingale $\hat{\varphi}^i \bullet \hat{S}^{i,\varepsilon}$ is of class D under $\hat{Q}^{i,\varepsilon}$ and hence a true $\hat{Q}^{i,\varepsilon}$-martingale. As $\frac{\mathrm{d}\hat{Q}^{i,\varepsilon}}{\mathrm{d}\hat{Q}^i} = \overline{Z}_T^{i,\varepsilon}$ is bounded, it suffices to show that $\hat{\varphi}^i \bullet \hat{S}^{i,\varepsilon}$ is of class D under $\hat{Q}^i$. By definition of $\hat{S}^{i,\varepsilon}$, an integration by parts (recall that $\hat{\varphi}^i$

[31] As φ is làdlàg (because it is of finite variation), integration by parts is well defined, and as $\Delta S^{i,\varepsilon}$ is continuous, we need not consider left and right limits.

is an Itô process) and a similar estimate as in (A.40), for $t \in [0,T]$, we obtain

$$\begin{aligned} |\hat{\varphi}^i \bullet \hat{S}^{i,\varepsilon}_t| &\le |\hat{\varphi}^i \bullet S_t| + |\hat{\varphi}^i \bullet \Delta\hat{S}^{i,\varepsilon}_t| \\ &\le |\hat{\varphi}^i \bullet S_t| + |\hat{\varphi}^i_{t\wedge\tilde{\tau}^i_\varepsilon} \Delta S^{i,\varepsilon}_{t\wedge\tilde{\tau}^i_\varepsilon}| + \left| \int_0^{t\wedge\tilde{\tau}^i_\varepsilon} \Delta S^{i,\varepsilon}_u \,\mathrm{d}\hat{\varphi}^i_u \right|. \end{aligned} \tag{A.44}$$

It suffices to show that each summand on the right-hand side of (A.44) is of class D under $\hat{Q}^i$. For the first summand this is trivial as it is the modulus of a (uniformly integrable) $\hat{Q}^i$-martingale. For the second summand, using that $\Delta S^{i,\varepsilon} \in [-\varepsilon\alpha^i\mathfrak{E}, \varepsilon\alpha^i\mathfrak{E}]$, $\mathfrak{E} \le \varepsilon^{-\frac{1}{6}}$ on $[\![0,\tau^{\mathfrak{E}}_\varepsilon]\!]$, $\tilde{\tau}^i_\varepsilon \le \tau^{\mathfrak{E}}_\varepsilon$ and Lemma E.4 (with the universal constant C_1), this follows from

$$\|(\hat{\varphi}^i \Delta S^{i,\varepsilon})^{\tilde{\tau}^i_\varepsilon}\|_{\mathcal{S}^1(\hat{Q}^i)} \le \varepsilon^{\frac{5}{6}}\alpha^i \|\hat{\varphi}^i\|_{\mathcal{S}^1(\hat{Q}^i)} \le \varepsilon^{\frac{5}{6}}\alpha^i C_1 \left(|\hat{\varphi}^i_0| + \|\hat{\varphi}^i\|_{\mathcal{H}^1(\hat{Q}^i)} \right) < \infty.$$

For the third summand, this follows from (A.13).

To establish (A.43), we first note that by Bayes' theorem,

$$E^{\hat{Q}^{i,\varepsilon}}\left[\hat{\varphi}^i \bullet \Delta\hat{S}^{i,\varepsilon}_T\right] = E^{\hat{Q}^i}\left[(1+\Delta\hat{Z}^{i,\varepsilon}_T)(\hat{\varphi}^i \bullet \Delta\hat{S}^{i,\varepsilon}_T)\right].$$

As $|\Delta\hat{Z}^{i,\varepsilon}| \le \frac{1}{2}\varepsilon^{\frac{1}{12}}$, it suffices to show that

$$E^{\hat{Q}^i}\left[\hat{\varphi}^i \bullet \Delta\hat{S}^{i,\varepsilon}_T\right] = -\varepsilon^{\frac{2}{3}}\hat{Z}^i_0 E^{\hat{Q}^i}\left[\int_0^T \frac{\alpha^i\mathfrak{E}_s}{\Delta\overline{\varphi}^i_s}\Sigma^{\hat{\varphi}^i}_s \,\mathrm{d}s\right] + o(\varepsilon^{\frac{2}{3}}).$$

An integration by parts and $\Delta\hat{S}^{i,\varepsilon}_0 = 0$ give

$$\hat{\varphi}^i \bullet \Delta\hat{S}^{i,\varepsilon}_T = \hat{\varphi}_{\tilde{\tau}^i_\varepsilon}\Delta S^{i,\varepsilon}_{\tilde{\tau}^i_\varepsilon} - \Delta S^{i,\varepsilon} \bullet \hat{\varphi}^i_{\tilde{\tau}^i_\varepsilon} - \langle \hat{\varphi}^i, \Delta S^{i,\varepsilon}\rangle_{\tilde{\tau}^i_\varepsilon}. \tag{A.45}$$

Now, take the $\hat{Q}^i$-expectation of each summand on the right-hand side of (A.45). Taking into account Lemma A.6 and $\Delta S^{i,\varepsilon} \in [-\varepsilon\alpha^i\mathfrak{E}, \varepsilon\alpha^i\mathfrak{E}]$, we obtain

$$E^{\hat{Q}^i}\left[\hat{\varphi}^i_{\tilde{\tau}^i_\varepsilon}\Delta S^{i,\varepsilon}_{\tilde{\tau}^i_\varepsilon}\right] = o(\varepsilon^{\frac{2}{3}}). \tag{A.46}$$

Next, (A.13) and Jensen's inequality imply

$$E^{\hat{Q}^i}\left[\Delta S^{i,\varepsilon} \bullet \hat{\varphi}^i_{\tilde{\tau}^i_\varepsilon}\right] = O(\varepsilon^{\frac{5}{6}}). \tag{A.47}$$

Finally, by (A.16), Lemma E.2, and Lemma A.4:

$$\begin{aligned} E^{\hat{Q}^i}\left[\langle \hat{\varphi}^i, \Delta S^{i,\varepsilon}\rangle_{\tilde{\tau}^i_\varepsilon}\right] &= -E^{\hat{Q}^i}\left[\frac{3}{2}\varepsilon^{\frac{2}{3}}\left((N^{i,\varepsilon}_s)^2 - 1\right)\frac{\alpha^i\mathfrak{E}}{\Delta\overline{\varphi}^i}\Sigma^{\hat{\varphi}^i} \bullet I_T\right] + O(\varepsilon) \\ &= \varepsilon^{\frac{2}{3}} E^{\hat{Q}^i}\left[\frac{\alpha^i\mathfrak{E}}{\Delta\overline{\varphi}^i}\Sigma^{\hat{\varphi}^i} \bullet I_T\right] + o(\varepsilon^{\frac{2}{3}}). \end{aligned} \tag{A.48}$$

Putting (A.46) – (A.48) together yields (A.43). □

We now compute the expected discounted square of the dual martingale correction $\Delta\hat{Z}^{i,\varepsilon}_t$. This is used in the proof of Theorem 3.2 to compute the second-order term of the dual upper bound.

Lemma A.19. *Fix $\kappa > 0$ and assume that $\mathcal{X}_{N^i}(\hat{Q}^i) \in L^1(\hat{Q}^i)$ and $E^{\hat{Q}^i}\left[\int_0^T \frac{\mathfrak{E}_s}{\Delta\overline{\varphi}^i_s}\Sigma^{\hat{\varphi}^i}_s \,\mathrm{d}s\right] < \infty$. Then:*

$$E^{\hat{Q}^i}\left[\int_{(0,T]} \frac{(\Delta\hat{Z}^{i,\varepsilon}_t)^2}{\gamma^i B_t}\,\nu(\mathrm{d}t)\right] = \frac{1}{2}\varepsilon^{\frac{2}{3}} E^{\hat{Q}^i}\left[\int_0^T \frac{\alpha^i\mathfrak{E}_s}{\Delta\overline{\varphi}^i_s}\Sigma^{\hat{\varphi}^i}_s \,\mathrm{d}s\right] + o(\varepsilon^{\frac{2}{3}}).$$

Proof. An integration by parts (recalling that $A_{T+} = 0$, $\Delta\hat{Z}^{i,\varepsilon}_0 = 0$ and A is of finite variation), Itô's formula, and recalling that $\langle\Delta\hat{Z}^{i,\varepsilon}\rangle = (1 + \Delta\hat{Z}^{i,\varepsilon})^2(\lambda^{i,\varepsilon})^2\mathbf{1}_{[\![0,\tilde{\tau}_\varepsilon]\!]} \bullet I$, A is uniformly bounded and $\Delta\hat{Z}^{i,\varepsilon}$ is a bounded martingale give

$$\begin{aligned}
E^{\hat{Q}^i}\left[\int_0^T \frac{(\Delta\hat{Z}^{i,\varepsilon}_t)^2}{\gamma^i B_t}\,\nu(\mathrm{d}t)\right] &= E^{\hat{Q}^i}\left[-\int_0^T \frac{(\Delta\hat{Z}^{i,\varepsilon}_t)^2}{\gamma^i}\,\mathrm{d}A_{t+}\right] = E^{\hat{Q}^i}\left[\int_0^T \frac{A_t}{\gamma^i}\,\mathrm{d}(\Delta\hat{Z}^{i,\varepsilon}_t)^2\right] \\
&= E^{\hat{Q}^i}\left[2\int_0^T \frac{A_t}{\gamma^i}\Delta\hat{Z}^{i,\varepsilon}_t\,\mathrm{d}\Delta\hat{Z}^{i,\varepsilon}_t + \int_0^{\tilde{\tau}^i_\varepsilon} \frac{A_t}{\gamma^i}(1 + \Delta\hat{Z}^{i,\varepsilon}_t)^2(\lambda^{i,\varepsilon}_t)^2\,\mathrm{d}t\right] \\
&= E^{\hat{Q}^i}\left[\int_0^{\tilde{\tau}^i_\varepsilon} \frac{A_t}{\gamma^i}(1 + \Delta\hat{Z}^{i,\varepsilon}_t)^2(\lambda^{i,\varepsilon}_t)^2\,\mathrm{d}t\right].
\end{aligned}$$

Now using that $|\Delta\hat{Z}^{i,\varepsilon}| \leq \frac{1}{2}\varepsilon^{\frac{1}{12}}$, the claim follows from Lemmas A.14 and E.2 and dominated convergence. □

A.9 Proof of Theorem 3.2

Using the results and estimates established above, we now prove Theorem 3.2.

Additional notation and preliminary estimates. To ease notation, define

$$\begin{aligned}
\Delta c^{i,D,\varepsilon} &:= -\frac{\varepsilon\alpha^i\mathfrak{E}}{A} \bullet \|\hat{\varphi}^{i,\varepsilon}\|, \\
\Delta c^{i,I,\varepsilon} &:= \frac{\Delta\hat{\varphi}^{i,\varepsilon}}{A}\mathbf{1}_{[\![0,\hat{\tau}^i_\varepsilon]\!]} \bullet S, \\
\Delta c^{i,\tau,\varepsilon} &:= \frac{\ell^i - \hat{\varphi}^i}{A}\mathbf{1}_{]\!]\tau^i_\varepsilon,T]\!]} \bullet S,
\end{aligned} \tag{A.49}$$

as well as

$$\Delta c^{i,\varepsilon} := \Delta c^{i,D,\varepsilon} + \Delta c^{i,I,\varepsilon} + \Delta c^{i,D,\varepsilon}.$$

Then, $\Delta c^{i,D,\varepsilon}$, $\Delta c^{i,I,\varepsilon}$, and $\Delta c^{i,\tau,\varepsilon}$ correspond to the consumption correction due to direct transaction costs, indirect transaction costs, and liquidation costs, respectively; $\Delta c^{i,\varepsilon}$ is the total consumption correction.

We proceed to show that the consumption corrections $\Delta c^{i,D,\varepsilon}$ and $\Delta c^{i,I,\varepsilon}$ are uniformly bounded and of order $o(1)$ and that $(\Delta c^{i,\tau,\varepsilon})^*_T$ is square integrable under P. Define

$$\begin{aligned}
\Upsilon^{i,D,\varepsilon} &;= -\varepsilon\alpha^i\mathfrak{E} \bullet \|\hat{\varphi}^{i,\varepsilon}\|, \\
\Upsilon^{i,I,\varepsilon} &:= \Delta\hat{\varphi}^{i,\varepsilon}\mathbf{1}_{[\![0,\hat{\tau}^i_\varepsilon]\!]} \bullet S, \\
\Upsilon^{i,\tau,\varepsilon} &:= (\ell^i - \hat{\varphi}^i)\mathbf{1}_{]\!]\hat{\tau}^i_\varepsilon,T]\!]} \bullet S.
\end{aligned}$$

By the definitions of $\hat{\tau}^{i,D}_\varepsilon$ and $\hat{\tau}^{i,I}_\varepsilon$, and $\hat{\tau}^i_\varepsilon \le \tau^{i,D}_\varepsilon \wedge \tau^{i,I}_\varepsilon$,

$$\begin{aligned}\left|\Upsilon^{i,D,\varepsilon}_T\right| &= \int_0^{\hat{\tau}^i_\varepsilon} \varepsilon\alpha^i \mathfrak{E}_t \,\mathrm{d}\|\varphi^{i,\varepsilon}\|_t + \varepsilon\alpha^i \mathfrak{E}_{\hat{\tau}^i_\varepsilon} |\varphi^{i,\varepsilon}_{\hat{\tau}^i_\varepsilon} - \ell^i| \mathbf{1}_{\{\hat{\tau}^i_\varepsilon < T\}} \le \alpha^i \varepsilon^{\frac{2}{9}\frac{\kappa}{1+\kappa}} \le \varepsilon^{\frac{2}{9}\frac{\kappa}{1+\kappa}}, \\ \left|\Upsilon^{i,I,\varepsilon}_T\right| &= \left|\int_0^{\hat{\tau}_\varepsilon} \Delta\varphi^{i,\varepsilon}_t \,\mathrm{d}S_t\right| \le \varepsilon^{\frac{1}{9}\frac{\kappa}{1+\kappa}}.\end{aligned}$$

Since $\Upsilon^{D,\varepsilon}$ is nondecreasing and $\Upsilon^{i,I,\varepsilon}$ is a $\hat{Q}^i$-martingale, we obtain that $\left(\Upsilon^{i,D,\varepsilon}\right)^*_T \le \varepsilon^{\frac{2}{9}\frac{\kappa}{1+\kappa}}$ and $\left(\Upsilon^{i,I,\varepsilon}\right)^*_T \le \varepsilon^{\frac{1}{9}\frac{\kappa}{1+\kappa}}$. Moreover,

$$\left(\Upsilon^{i,\tau,\varepsilon}\right)^*_T \le 2\left((\ell^i - \hat{\varphi}^i) \bullet S\right)^*_T.$$

Hence, Lemma E.8 gives

$$\left(\Delta c^{i,D,\varepsilon}\right)^*_T \le \frac{2}{A_T}\left(\Upsilon^{i,D,\varepsilon}\right)^*_T \le \frac{2}{A_T}\varepsilon^{\frac{2}{9}\frac{\kappa}{1+\kappa}}, \tag{A.50}$$

$$\left(\Delta c^{i,I,\varepsilon}\right)^*_T \le \frac{2}{A_T}\left(\Upsilon^{i,I,\varepsilon}\right)^*_T \le \frac{2}{A_T}\varepsilon^{\frac{1}{9}\frac{\kappa}{1+\kappa}}, \tag{A.51}$$

$$\left(\Delta c^{i,\tau,\varepsilon}\right)^*_T \le \frac{2}{A_T}\left(\Upsilon^{i,\tau,\varepsilon}\right)^*_T \le \frac{4}{A_T}\left((\ell^i - \hat{\varphi}^i) \bullet S\right)^*_T. \tag{A.52}$$

Moreover, note that by $\mathcal{X}_{\hat{\varphi}^i}(P) \in L^2(P)$ and Lemma E.4.

$$E\left[\left(\left((\ell^i - \hat{\varphi}^i) \bullet S\right)^*_T\right)^2\right] < \infty. \tag{A.53}$$

Admissibility of candidate strategy/consumption pair. We proceed to show that the candidate strategy/consumption pair $(\hat{\varphi}^{i,\varepsilon}, \hat{c}^{i,\varepsilon})$ is in $\mathcal{A}_{\varepsilon^i}(\hat{\varphi}^i_0, Y^i)$. By construction, $\hat{\varphi}^{i,\varepsilon}_0 = \varphi^{i,\varepsilon}_0 = \hat{\varphi}^i_0$. Moreover, $\hat{\tau}^i_\varepsilon \le \tau^{i,I}_\varepsilon$ implies $|(\Delta\varphi^{i,\varepsilon} \bullet S)^{\hat{\tau}^i_\varepsilon}| \le \varepsilon^{\frac{1}{9}\frac{\kappa}{1+\kappa}}$ and yields

$$\begin{aligned}\hat{\varphi}^{i,\varepsilon} \bullet S &= (\hat{\varphi}^i \bullet S)^{\hat{\tau}^i_\varepsilon} + (\Delta\varphi^{i,\varepsilon} \bullet S)^{\hat{\tau}^i_\varepsilon} + \ell^i(S - S^{\hat{\tau}^i_\varepsilon}) \\ &\ge (\hat{\varphi}^i \bullet S)^{\hat{\tau}^i_\varepsilon} - \varepsilon^{\frac{1}{9}\frac{\kappa}{1+\kappa}} + \ell^i(S - S^{\hat{\tau}^i_\varepsilon}).\end{aligned}$$

Hence, for any $Q \in \mathcal{M}^2(S)$, the local Q-martingale $\hat{\varphi}^{i,\varepsilon} \bullet S$ is bounded from below by the Q-supermartingale $(\hat{\varphi}^i \bullet S)^{\hat{\tau}^i_\varepsilon} + \ell^i(S - S^{\hat{\tau}^i_\varepsilon}) - \varepsilon^{\frac{1}{9}\frac{\kappa}{1+\kappa}}$. Hence, it also is a Q-supermartingale.

Moreover, by feasibility of $\hat{c}^i$, (A.50)-(A.52) and the estimate (A.53), we obtain

$$E\left[\left(\int_0^T |\hat{c}^{i,\varepsilon}_t| \,\nu(\mathrm{d}t)\right)^2\right] \le 2E\left[\left(\int_0^T |\hat{c}^i_t| \,\nu(\mathrm{d}t)\right)^2\right] + 2E\left[\left(\int_0^T |\Delta c^{i,\varepsilon}_t| \,\nu(\mathrm{d}t)\right)^2\right] < \infty.$$

Finally, the definition of $\hat{c}^{i,\varepsilon}$, Lemma E.8 and the fact that $X_T^{\hat{\varphi}^i,\hat{c}^i} = 0$ give

$$
\begin{aligned}
X_T^{\hat{\varphi}^{i,\varepsilon},\hat{c}^{i,\varepsilon},\varepsilon^i} &= \int_0^T \hat{\varphi}_t^{i,\varepsilon}\,\mathrm{d}S_t - \int_0^T \varepsilon_t^i\,\mathrm{d}\|\hat{\varphi}^{i,\varepsilon}\|_t - \int_0^T \frac{\hat{c}^{i,\varepsilon}}{B_t}\,\nu(\mathrm{d}t) + Y_T^i \\
&= \int_0^T \hat{\varphi}_t^i\,\mathrm{d}S_t - \int_0^T (\hat{\varphi}_t^i - \hat{\varphi}_t^{i,\varepsilon})\,\mathrm{d}S_t - \int_0^T \alpha^i\varepsilon\mathfrak{E}_t\,\mathrm{d}\|\hat{\varphi}^{i,\varepsilon}\|_t \\
&\quad - \int_0^T \frac{\hat{c}}{B_t}\,\nu(\mathrm{d}t) - \int_0^T \frac{\Delta c^{i,\varepsilon}}{B_t}\,\nu(\mathrm{d}t) + Y_T^i \\
&= X_T^{\hat{\varphi},\hat{c}} + 0 = 0.
\end{aligned}
$$

In summary, $(\hat{\varphi}^{i,\varepsilon}, \hat{c}^{i,\varepsilon}) \in \mathcal{A}_{\varepsilon^i}(\hat{\varphi}_0^i, Y^i)$ as claimed.

Primal lower bound. Next, we establish the asymptotic formula (3.8) for the performance of our candidate policy. Note that in view of (3.9), it suffices to prove this with "=" replaced by "≥". Let $\varepsilon \in (0,1)$. For fixed t and ω, a Taylor expansion of order two with Lagrange remainder term gives

$$
U_t^i\left(\hat{c}_t^{i,\varepsilon}\right) = U_t^i(\hat{c}_t^i) + U_t^{i\prime}(\hat{c}_t^i)\left(\hat{c}_t^{i,\varepsilon} - \hat{c}_t^i\right) + \frac{1}{2}U_t^{i\prime\prime}(\hat{c}_t^i)\left(\hat{c}_t^{i,\varepsilon} - \hat{c}_t^i\right)^2 + \frac{1}{6}U_t^{i\prime\prime\prime}(\tilde{c}(t,\omega))\left(\hat{c}_t^{i,\varepsilon} - \hat{c}_t^i\right)^3,
$$

where $\tilde{c}(t,\omega)$ takes values in the interval with endpoints $\hat{c}_t^i(\omega)$ and $\hat{c}_t^{i,\varepsilon}(\omega)$. Since $U^{i\prime}$ is nonincreasing and $U^{i\prime\prime\prime} = -\gamma^i U^{i\prime\prime} = (\gamma^i)^2 U^{i\prime}$, we obtain

$$
U_t^i\left(\hat{c}_t^{i,\varepsilon}\right) \geq U_t^i(\hat{c}_t) + U_t^{i\prime}(\hat{c}_t^i)\left(\Delta c^{i,\varepsilon} - \frac{\gamma^i}{2}\left(\Delta c^{i,\varepsilon}\right)^2 - \frac{(\gamma^i)^2}{6}\exp\left(\gamma^i|\Delta c_t^{i,\varepsilon}|\right)|\Delta c^{i,\varepsilon}|^3\right). \tag{A.54}
$$

We proceed to derive a lower bound for

$$
-\frac{\gamma^i}{2}\left(\Delta c^{i,\varepsilon}\right)^2 - \frac{(\gamma^i)^2}{6}\exp\left(\gamma^i|\Delta c_t^{i,\varepsilon}|\right)|\Delta c^{i,\varepsilon}|^3. \tag{A.55}
$$

First, consider (A.55) on the event $\{\hat{\tau}_\varepsilon^i = T\}$. Then $\Delta c^{i,\tau,\varepsilon} = 0$ and $\Delta c^{i,\varepsilon} = \Delta c^{i,D,\varepsilon} + \Delta c^{i,I,\varepsilon}$. Using that $|\Delta c^{i,D,\varepsilon}|, |\Delta c^{i,I,\varepsilon}| \leq \frac{2}{A_T}\varepsilon^{\frac{1}{9}\frac{\kappa}{1+\kappa}}$ by (A.50) and (A.51) gives

$$
\begin{aligned}
&\left(-\frac{\gamma^i}{2}\left(\Delta c^{i,\varepsilon}\right)^2 - \frac{(\gamma^i)^2}{6}\exp\left(\gamma^i|\Delta c_t^{i,\varepsilon}|\right)|\Delta c^{i,\varepsilon}|^3\right)\mathbf{1}_{\{\hat{\tau}_\varepsilon = T\}} \\
&\quad \geq -\frac{\gamma^i}{2}\left(\Delta c^{i,D,\varepsilon} + \Delta c^{i,I,\varepsilon}\right)^2\left(1 + \frac{\gamma^i}{3}\exp\left(\frac{4}{A_T}\gamma^i\right)\frac{4}{A_T}\varepsilon^{\frac{1}{9}\frac{\kappa}{1+\kappa}}\right)\mathbf{1}_{\{\hat{\tau}_\varepsilon = T\}}.
\end{aligned} \tag{A.56}
$$

Since $\Delta c^{i,D,\varepsilon}$ is nonpositive and $|\Delta c^{i,D,\varepsilon}| + 2|\Delta c^{i,I,\varepsilon}| \leq \frac{6}{A_T}\varepsilon^{\frac{1}{9}\frac{\kappa}{1+\kappa}}$ by (A.50) and (A.51) , we have the following estimate:

$$
\begin{aligned}
(\Delta c^{i,D,\varepsilon} + \Delta c^{i,I,\varepsilon})^2 &\leq (\Delta c^{i,I,\varepsilon})^2 - \Delta c^{i,D,\varepsilon}\left(|\Delta c^{i,D,\varepsilon}| + 2|\Delta c^{i,I,\varepsilon}|\right) \\
&\leq (\Delta c^{i,I,\varepsilon})^2 - \Delta c^{i,D,\varepsilon}\frac{6}{A_T}\varepsilon^{\frac{1}{9}\frac{\kappa}{1+\kappa}}.
\end{aligned}
$$

Plugging this into (A.56), setting $C_2 := \frac{4}{3}\frac{\gamma^i}{A_T}\exp\left(\frac{4}{A_T}\gamma^i\right)$ and using that $\mathbf{1}_{\{\hat{\tau}^i_\varepsilon = T\}} \leq 1$ gives

$$\begin{aligned}
&\left(-\frac{\gamma^i}{2}\left(\Delta c^{i,\varepsilon}\right)^2 - \frac{(\gamma^i)^2}{6}\exp\left(\gamma^i\left|\Delta c^{i,\varepsilon}_t\right|\right)\left|\Delta c^{i,\varepsilon}\right|^3\right)\mathbf{1}_{\{\hat{\tau}^i_\varepsilon = T\}} \\
&\qquad \geq -\frac{\gamma^i}{2}\left((\Delta c^{i,I,\varepsilon})^2 - \frac{6}{A_T}\varepsilon^{\frac{1}{9}\frac{\kappa}{1+\kappa}}\Delta c^{i,D,\varepsilon}\right)\left(1 + C_2\varepsilon^{\frac{1}{9}\frac{\kappa}{1+\kappa}}\right). \qquad \text{(A.57)}
\end{aligned}$$

Next, consider (A.55) on the event $\{\hat{\tau}^i_\varepsilon < T\}$. The series expansion of the exponential function implies that, for $x \geq 0$,

$$\begin{aligned}
\exp\left(\gamma^i\frac{\kappa}{2}x\right) &\geq \frac{1}{2}\left(\gamma^i\frac{\kappa}{2}x\right)^2 + \frac{1}{6}\left(\gamma^i\frac{\kappa}{2}x\right)^3 = \frac{\gamma^i}{2}x^2\frac{\gamma^i\kappa^2}{4} + \frac{(\gamma^i)^2}{6}x^3\frac{\kappa^3\gamma^i}{8} \\
&\geq \frac{\gamma^i\kappa^2}{8}\min(1,\kappa)\left(\frac{\gamma^i}{2}x^2 + \frac{(\gamma^i)^2}{6}x^3\right).
\end{aligned}$$

Whence, for $x \geq 0$:

$$\begin{aligned}
\frac{\gamma^i}{2}x^2 + \frac{(\gamma^i)^2}{6}\exp\left(\gamma^i x\right)x^3 &\leq \exp\left(\gamma^i x\right)\left(\frac{\gamma^i}{2}x^2 + \frac{(\gamma^i)^2}{6}x^3\right) \\
&\leq \frac{8}{\gamma^i\kappa^2\min(1,\kappa)}\exp\left(\gamma^i x\right)\exp\left(\gamma^i\frac{\kappa}{2}x\right) \\
&= \frac{8}{\gamma^i\kappa^2\min(1,\kappa)}\exp\left(\gamma^i\left(1 + \frac{\kappa}{2}\right)x\right).
\end{aligned}$$

Using that $|\Delta c^{i,D,\varepsilon}|, |\Delta c^{i,I,\varepsilon}| \leq \frac{2}{A_T}\varepsilon^{\frac{1}{9}\frac{\kappa}{1+\kappa}}$ by (A.50) and (A.51) it follows that

$$\begin{aligned}
&\left(-\frac{\gamma^i}{2}\left(\Delta c^{i,\varepsilon}\right)^2 - \frac{(\gamma^i)^2}{6}\exp\left(\gamma^i\left|\Delta c^{i,\varepsilon}_t\right|\right)\left|\Delta c^{i,\varepsilon}\right|^3\right)\mathbf{1}_{\{\hat{\tau}^i_\varepsilon < T\}} \\
&\qquad \geq -\frac{8}{\gamma^i\kappa^2\min(1,\kappa)}\exp\left(\gamma^i\left(1 + \frac{\kappa}{2}\right)\left|\Delta c^{i,\varepsilon}_t\right|\right)\mathbf{1}_{\{\hat{\tau}^i_\varepsilon < T\}} \\
&\qquad \geq -C_3\exp\left(\gamma^i\left(1 + \frac{\kappa}{2}\right)\left|\Delta c^{i,\tau,\varepsilon}_t\right|\right)\mathbf{1}_{\{\hat{\tau}^i_\varepsilon < T\}}, \qquad \text{(A.58)}
\end{aligned}$$

where

$$C_3 := \frac{8}{\gamma^i\kappa^2\min(1,\kappa)}\exp\left(\frac{4}{A_T}\gamma^i\left(1 + \frac{\kappa}{2}\right)\right).$$

Plugging (A.57) and (A.58) into (A.54) gives

$$\begin{aligned}
U^i_t(\hat{c}^{i,\varepsilon}_t) - U^i_t(\hat{c}^i_t) \geq U^{i\prime}_t(\hat{c}_t)\Bigg(&\left(\Delta c^{i,D,\varepsilon}_t + \Delta c^{i,I,\varepsilon}_t + \Delta c^{i,\tau,\varepsilon}_t\right) \\
&+ \frac{3\gamma^i}{A_T}\Delta c^{i,D,\varepsilon}_t\left(\varepsilon^{\frac{1}{9}\frac{\kappa}{1+\kappa}} + C_2\varepsilon^{\frac{2}{9}\frac{\kappa}{1+\kappa}}\right) \\
&- \frac{\gamma^i}{2}\left(\Delta c^{i,I,\varepsilon}_t\right)^2\left(1 + C_2\varepsilon^{\frac{1}{9}\frac{\kappa}{1+\kappa}}\right) \\
&- \mathbf{1}_{\{\hat{\tau}_\varepsilon < T\}}C_3\exp\left(\gamma^i\left(1 + \frac{\kappa}{2}\right)\left|\Delta c^{i,\tau,\varepsilon}_t\right|\right)\Bigg). \qquad \text{(A.59)}
\end{aligned}$$

We now calculate the expectation of the ν-integral of the all the terms on the right-hand side of

(A.59). To this end, we use the first-order condition $U_t^{i\prime}(\hat{c}_t^i) = \frac{\hat{Z}_t^i}{B_t}$.

First, by Bayes' theorem [34, Corollary A.2(b)] and Lemmas E.8, A.9, E.2 and A.6, we obtain

$$\begin{aligned} E\left[\int_0^T U_t^{i\prime}(\hat{c}_t^i)\Delta c_t^{i,D,\varepsilon}\,\nu(\mathrm{d}t)\right] &= E^{\hat{Q}^i}\left[\int_0^T \frac{\Delta c_t^{i,D,\varepsilon}}{B_t}\,\nu(\mathrm{d}t)\right] = E^{\hat{Q}^i}\left[\Upsilon_T^{i,D,\varepsilon}\right] \\ &= E^{\hat{Q}^i}\left[-\int_0^{\hat{\tau}_\varepsilon^i} \varepsilon\alpha^i \mathfrak{E}_s\,\mathrm{d}\|\varphi^{i,\varepsilon}\|_s - \varepsilon\alpha^i\mathfrak{E}_{\hat{\tau}_\varepsilon^i}|\varphi_{\hat{\tau}_\varepsilon^i}^{i,\varepsilon} - \ell^i|\mathbf{1}_{\{\hat{\tau}_\varepsilon^i<T\}}\right] \\ &= -\frac{1}{2}\varepsilon^{\frac{2}{3}}E^{\hat{Q}^i}\left[\int_0^T \frac{\alpha^i\mathfrak{E}_s}{\Delta\overline{\varphi}_s^i}\Sigma_s^{\hat{\varphi}^i}\,\mathrm{d}s\right] + o(\varepsilon^{\frac{2}{3}}). \end{aligned} \tag{A.60}$$

Second, Bayes' theorem [34, Corollary A.2(a)], Lemma E.8 and the $\hat{Q}^i$-martingale property of $\Upsilon^{i,I,\varepsilon}$ and $\Upsilon^{i,\tau,\varepsilon}$ give

$$E\left[\int_0^T U_t^{i\prime}(\hat{c}_t^i)\Delta c_t^{i,I,\varepsilon}\,\nu(\mathrm{d}t)\right] = E^{\hat{Q}^i}\left[\int_0^T \frac{\Delta c_t^{i,I,\varepsilon}}{B_t}\,\nu(\mathrm{d}t)\right] = E^{\hat{Q}^i}\left[\Upsilon_T^{i,I,\varepsilon}\right] = 0, \tag{A.61}$$

$$E\left[\int_0^T U_t^{i\prime}(\hat{c}_t^i)\Delta c_t^{i,\tau,\varepsilon}\,\nu(\mathrm{d}t)\right] = E^{\hat{Q}^i}\left[\int_0^T \frac{\Delta c_t^{i,\tau,\varepsilon}}{B_t}\,\nu(\mathrm{d}t)\right] = E^{\hat{Q}^i}\left[\Upsilon_T^{i,\tau,\varepsilon}\right] = 0. \tag{A.62}$$

Third, it follows from (A.60) that

$$E\left[\int_0^T U_t^{i\prime}(\hat{c}_t^i)\frac{3\gamma^i}{A_T}\Delta c_t^{i,D,\varepsilon}\left(\varepsilon^{\frac{1}{9}\frac{\kappa}{1+\kappa}} + C_2\varepsilon^{\frac{2}{9}\frac{\kappa}{1+\kappa}}\right)\nu(\mathrm{d}t)\right] = o(\varepsilon^{\frac{3}{2}}). \tag{A.63}$$

Fourth, we proceed to show that

$$\begin{aligned} E\left[\int_0^T U_t^{i\prime}(\hat{c}_t^i)\left(-\frac{\gamma^i}{2}\left(\Delta c_t^{i,I,\varepsilon}\right)^2\left(1 + C_2\varepsilon^{\frac{1}{9}\frac{\kappa}{1+\kappa}}\right)\right)\nu(\mathrm{d}t)\right] \\ = -\frac{1}{4}\varepsilon^{\frac{2}{3}}E^{\hat{Q}^i}\left[\int_0^T \frac{\alpha^i\mathfrak{E}_t}{\Delta\overline{\varphi}_t^i}\Sigma_t^{\hat{\varphi}^i}\,\mathrm{d}t\right] + o(\varepsilon^{\frac{2}{3}}). \end{aligned} \tag{A.64}$$

To this end, it suffices to show that

$$E^{\hat{Q}^i}\left[\int_0^T \gamma^i\frac{\left(\Delta c_t^{i,I,\varepsilon}\right)^2}{B_t}\,\nu(\mathrm{d}t)\right] = \frac{1}{2}\varepsilon^{\frac{2}{3}}E^{\hat{Q}^i}\left[\int_0^T \frac{\alpha^i\mathfrak{E}_t}{\Delta\overline{\varphi}_t^i}\Sigma_t^{\hat{\varphi}^i}\,\mathrm{d}t\right] + o(\varepsilon^{\frac{2}{3}}). \tag{A.65}$$

An integration by parts (recalling that $A_{T+} =: 0$ and $\Delta c_0^{i,I,\varepsilon} = 0$), the definition of $\Delta c^{i,I,\varepsilon}$, Itô's formula and the identity $\frac{\gamma^i}{A}(\Delta\varphi^{i,\varepsilon})^2\Sigma^S = \frac{3}{2}\varepsilon^{\frac{2}{3}}(N^{i,\varepsilon})^2\frac{\alpha^i\mathfrak{E}}{\Delta\overline{\varphi}^i}\Sigma^{\hat\varphi^i}$ give

$$\begin{aligned}\int_0^T \frac{\gamma^i(\Delta c_t^{i,I,\varepsilon})^2}{B_t}\,\nu(\mathrm{d}t) &= -\int_0^T \gamma^i\left(\Delta c_t^{i,I,\varepsilon}\right)^2 \mathrm{d}A_t = \int_0^T \gamma^i A_t\,\mathrm{d}\left(\Delta c_t^{i,I,\varepsilon}\right)^2\\
&= 2\gamma^i\int_0^T \Upsilon_{t-}^{i,I,\varepsilon}\,\mathrm{d}\Upsilon_t^{i,I,\varepsilon} + \int_0^T \frac{\gamma^i}{A_t}\,\mathrm{d}[\Upsilon^{i,I,\varepsilon}]_t\\
&= 2\gamma^i\int_0^T \Upsilon_{t-}^{i,I,\varepsilon}\,\mathrm{d}\Upsilon_t^{i,I,\varepsilon} + \int_0^{\hat\tau_\varepsilon^i} \frac{\gamma^i}{A_t}\left(\Delta\varphi^{i,\varepsilon}\right)^2\Sigma_t^S\,\mathrm{d}t\\
&= 2\int_0^T \Upsilon_{t-}^{i,I,\varepsilon}\,\mathrm{d}\Upsilon_t^{i,I,\varepsilon} + \frac{3}{2}\varepsilon^{\frac{2}{3}}\int_0^{\hat\tau_\varepsilon}(N_t^{i,\varepsilon})^2\frac{\alpha^i\mathfrak{E}_t}{\Delta\overline{\varphi}_t^i}\Sigma_t^{\hat\varphi^i}\,\mathrm{d}t.\end{aligned}$$

As $\Upsilon_-^{i,I,\varepsilon}\bullet\Upsilon^{i,I,\varepsilon}$ is a $\hat{Q}$-martingale (because $\Upsilon^{i,I,\varepsilon}$ is a bounded $\hat{Q}$-martingale), (A.65) follows from Lemmas A.4 and E.2.

Finally, Bayes' theorem [34, Corollary A.2.(a)], the fact that $1/B$ is uniformly bounded from above, Hölder's inequality with $p = 1+\frac{\kappa}{2}$ and $q = 1+\frac{2}{\kappa}$, Lemma E.6, Assumption A.2 (with $\tilde{C}_3 := C_3(\nu((0,T])\left(\frac{1}{B}\right)_T^*)$ together with Lemmas A.10 and A.11 show

$$\begin{aligned}&E\left[\int_0^T U_t^{i\prime}(\hat{c}_t^i)\left(-\mathbf{1}_{\{\hat\tau_\varepsilon^i<T\}}C_3\exp\left(\gamma^i\left(1+\frac{\kappa}{2}\right)|\Delta c_t^{i,\tau,\varepsilon}|\right)\right)\nu(\mathrm{d}t)\right]\\
&\quad\le \tilde{C}_3E^{\hat{Q}^i}\left[\mathbf{1}_{\{\tau_\varepsilon^i<T\}}\exp\left(\gamma^i\left(1+\frac{\kappa}{2}\right)(\Delta c^{i,\tau,\varepsilon})_T^*\right)\right]\\
&\quad\le \tilde{C}_3\hat{Q}^i[\hat\tau_\varepsilon^i<T]^{\frac{1}{1+\frac{\kappa}{2}}}E^{\hat{Q}^i}\left[\exp\left(\gamma^i\left(1+\frac{\kappa}{2}\right)\left(1+\frac{2}{\kappa}\right)\frac{\hat\varphi^i-\ell^i}{A}\mathbf{1}_{]\!]\hat\tau_\varepsilon^i,T]\!]}\bullet S\right)_T^*\right)\right]^{\frac{1}{1+\frac{2}{\kappa}}}\\
&\quad\le \tilde{C}_3\hat{Q}^i[\hat\tau_\varepsilon^i<T]^{\frac{1}{1+\frac{\kappa}{2}}}\left(8E^{\hat{Q}^i}\left[\exp\left(2(\gamma^i)^2\left(1+\frac{\kappa}{2}\right)^2\left(1+\frac{2}{\kappa}\right)^2\frac{(\hat\varphi^i-\ell^i)^2}{A^2}\Sigma^S\mathbf{1}_{]\!]\hat\tau_\varepsilon^i,T]\!]}\bullet I_T\right)\right]\right)^{\frac{1}{1+\frac{2}{\kappa}}}\\
&\quad\le \tilde{C}_3\hat{Q}^i[\hat\tau_\varepsilon^i<T]^{\frac{1}{1+\frac{\kappa}{2}}}\left(8E^{\hat{Q}^i}\left[\exp\left(2(\gamma^i)^2\left(2+\frac{\kappa}{2}+\frac{2}{\kappa}\right)^2\frac{(\hat\varphi^i-\ell^i)^2}{A^2}\Sigma^S\bullet I_T\right)\right]\right)^{\frac{1}{1+\frac{2}{\kappa}}}\\
&\quad= o(\varepsilon^{\frac{2}{3}}).\end{aligned}\tag{A.66}$$

Combining (A.60) – (A.66) yields (3.8) with "$\geq$".

Dual upper bound. We proceed to establish the asymptotic upper bound (3.9) for any admissible policy. To this end, note that in view of (3.8), it suffices to prove this with "=" replaced by "$\geq$". Fix $\varepsilon\in(0,1)$. Consider any admissible strategy/consumption pair $(\varphi^\varepsilon, c^\varepsilon)\in\mathcal{A}_{\varepsilon^i}(\hat\varphi_0^i, Y^i)$ with $E\left[\int_0^T U_t^{i-}(c_t^\varepsilon)\,\nu(\mathrm{d}t)\right]<\infty$. We want to establish an upper bound for $U_t^i(c_t^\varepsilon)$. To this end, define the martingale $(\hat{Z}_t^{i,\varepsilon})_{t\in[0,T]}$ by

$$\hat{Z}_t^{i,\varepsilon} = \hat{Z}_t^i\overline{Z}_t^{i,\varepsilon} = \hat{Z}_t(1+\Delta\hat{Z}_t^{i,\varepsilon}),\quad t\in[0,T].$$

By definition of the conjugate $\tilde{U}^i(y) = \sup_{x\in\mathbb{R}}(U^i(x)-xy)$, we have the pointwise inequality

$$U_t^i(c_t^\varepsilon)\le\tilde{U}_t^i\left(\frac{\hat{Z}_t^{i,\varepsilon}}{B_t}\right)+\frac{\hat{Z}_t^{i,\varepsilon}}{B_t}c_t^\varepsilon.\tag{A.67}$$

We proceed to establish an upper bound for $\tilde{U}_t^i\left(\frac{\hat{Z}_t^{i,\varepsilon}}{B_t}\right)$. For fixed t and ω, a Taylor expansion of order two with Lagrange remainder term gives

$$\begin{aligned}\tilde{U}_t^i\left(\frac{\hat{Z}_t^{i,\varepsilon}}{B_t}\right) &= \tilde{U}_t^i\left(\frac{\hat{Z}_t^i}{B_t}\right) + \tilde{U}_t^{i\prime}\left(\frac{\hat{Z}_t^i}{B_t}\right)\frac{\hat{Z}_t^i}{B_t}\Delta\hat{Z}_t^{i,\varepsilon} + \frac{1}{2}\tilde{U}_t^{i\prime\prime}\left(\frac{\hat{Z}_t^i}{B_t}\right)\frac{\left(\hat{Z}_t^i\right)^2}{B_t^2}(\Delta\hat{Z}_t^{i,\varepsilon})^2 \\ &\quad + \frac{1}{6}\tilde{U}_t^{i\prime\prime\prime}(\zeta(t,\omega))\frac{\left(\hat{Z}_t^i\right)^3}{B_t^3}(\Delta\hat{Z}_t^{i,\varepsilon})^3, \end{aligned} \tag{A.68}$$

where $\zeta(t,\omega)$ lies in the interval with endpoints $\frac{\hat{Z}_t^i(\omega)}{B_t(\omega)}$ and $\frac{\hat{Z}_t^{i,\varepsilon}(\omega)}{B_t(\omega)}$. Using $-\tilde{U}^{i\prime} = (U^{i\prime})^{-1}$ together with the first-order condition $U'(\hat{c}_t^i) = \frac{\hat{Z}_t^i}{B_t}$ and the fact that $U^i(x) = -\exp(-\gamma^i x)$ gives $\tilde{U}_t^{i\prime}\left(\frac{\hat{Z}_t^i}{B_t}\right) = -\hat{c}_t^i$, $\tilde{U}_t^{i\prime\prime}\left(\frac{\hat{Z}_t^i}{B_t}\right) = \frac{1}{\gamma^i}\frac{B_t}{\hat{Z}_t^i}$ and $\tilde{U}_t^{i\prime\prime\prime}(\zeta) = -\frac{1}{\gamma^i\zeta^2}$. Plugging this into (A.68) and using that $\hat{Z}_t^{i,\varepsilon} \geq \frac{\hat{Z}_t^i}{2}$ gives

$$\tilde{U}_t^i\left(\frac{\hat{Z}_t^{i,\varepsilon}}{B_t}\right) \leq \tilde{U}_t^i\left(\frac{\hat{Z}_t^i}{B_t}\right) - \hat{Z}_t^i\Delta\hat{Z}_t^{i,\varepsilon}\frac{\hat{c}_t^i}{B_t} + \frac{1}{2}\hat{Z}_t^i\frac{\left(\Delta\hat{Z}_t^{\varepsilon}\right)^2}{\gamma^i B_t} + \frac{2}{3}\hat{Z}_t^i\frac{\left|\Delta\hat{Z}_t^{i,\varepsilon}\right|^3}{\gamma^i B_t}. \tag{A.69}$$

By definition of $\tilde{U}^i$, the fact that U^i is increasing and C^1 on $\mathbb{R}$, and the first-order condition (3.3), we obtain

$$\tilde{U}^i\left(\frac{\hat{Z}_t^i}{B_t}\right) = U^i\left((U^{i\prime})^{-1}\left(\frac{\hat{Z}_t^i}{B_t}\right)\right) - (U^{i\prime})^{-1}\left(\frac{\hat{Z}_t^i}{B_t}\right)\frac{\hat{Z}_t^i}{B_t} = U_t^i(\hat{c}_t^i) - \hat{c}_t^i\frac{\hat{Z}_t^i}{B_t}. \tag{A.70}$$

Now plugging (A.69) and (A.70) into (A.67) yields

$$U_t^i(c_t^\varepsilon) - U_t^i(\hat{c}_t^i) \leq \frac{\hat{Z}_t^{i,\varepsilon}}{B_t}(c_t^\varepsilon - \hat{c}_t^i) + \frac{1}{2}\hat{Z}_t^i\frac{\left(\Delta\hat{Z}_t^{i,\varepsilon}\right)^2}{\gamma^i B_t}\left(1 + \frac{4}{3}\left|\Delta\hat{Z}_t^{i,\varepsilon}\right|\right). \tag{A.71}$$

We proceed to calculate the expectation of the ν-integral of both terms on the right-hand side of (A.71). First, by Bayes' theorem [34, Corollary A.2.(b)], the budget constraint (2.10) for c^ε, the budget constraint (2.6) for $\hat{c}$ (which is binding), and Lemmas A.17 and A.18, we obtain

$$\begin{aligned} E\left[\int_0^T \hat{Z}_t^{i,\varepsilon}\frac{c_t^\varepsilon - \hat{c}_t^i}{B_t}\,\nu(\mathrm{d}t)\right] &= E\left[\hat{Z}_T^{i,\varepsilon}\int_0^T \frac{c_t^\varepsilon - \hat{c}_t^i}{B_t}\,\nu(\mathrm{d}t)\right] \\ &\leq \hat{Z}_0^i E^{\hat{Q}^{i,\varepsilon}}\left[\varphi^\varepsilon \bullet S_T - \varepsilon\alpha^i\mathfrak{E}\bullet\|\varphi^\varepsilon\|_T + Y_T^i - \hat{\varphi}^i\bullet S_T - Y_T^i\right] \\ &\leq o(\varepsilon^{\frac{2}{3}}) - \varepsilon^{\frac{2}{3}}\hat{Z}_0^i E^{\hat{Q}^i}\left[\int_0^T \frac{\alpha^i\mathfrak{E}_s}{\Delta\overline{\varphi}_s^i}\Sigma_s^{\hat{\varphi}^i}\,\mathrm{d}s\right]. \end{aligned} \tag{A.72}$$

Next, Bayes' theorem [34, Corollary A.2.(b)]), Lemma A.19 and $|\Delta Z^{i,\varepsilon}| \le \frac{1}{2}\varepsilon^{\frac{1}{12}}$ give

$$
\begin{aligned}
&\frac{1}{2}E\left[\int_0^T \hat{Z}_t^i \frac{(\Delta\hat{Z}_t^{i,\varepsilon})^2}{\gamma^i B_t}\left(1+\frac{4}{3}|\Delta\hat{Z}_t^{i,\varepsilon}|\right)\nu(\mathrm{d}t)\right] \\
&\quad = \frac{1}{2}\hat{Z}_0^i E^{\hat{Q}^i}\left[\int_{(0,T]} \frac{1}{\gamma^i B_t}(\Delta\hat{Z}_t^{i,\varepsilon})^2\left(1+\frac{4}{3}|\Delta\hat{Z}_t^{i,\varepsilon}|\right)\nu(\mathrm{d}t)\right] \\
&\quad = \frac{1}{4}\varepsilon^{\frac{2}{3}}\hat{Z}_0^i E^{\hat{Q}^i}\left[\int_0^T \frac{\alpha^i \mathfrak{E}_s}{\Delta\overline{\varphi}_s^i}\Sigma_s^{\hat{\varphi}^i}\,\mathrm{d}s\right] + o(\varepsilon^{\frac{2}{3}}).
\end{aligned}
\tag{A.73}
$$

Combining (A.72) and (A.73) then finally yields (3.9) (with "=" replaced by "$\ge$".) This completes the proof of Theorem 3.2.

A.10 Proof of Theorem 3.4

We apply [34, Theorem 4.1 and Remark 4.2]. To this end, we need to show that the candidate consumption rate $\hat{c}^{3,\varepsilon}$ is feasible (with binding budget constraint). Noting that

$$\hat{c}^{3,\varepsilon} := \hat{c}^3 + \Delta c^{3,\varepsilon} = -\Delta c^{1,D,\varepsilon} - \Delta c^{2,D,\varepsilon},$$

with $\Delta c^{1,D,\varepsilon}$ and $\Delta c^{2,D,\varepsilon}$ as in (A.49), it follows from feasibility of $\hat{c}^3$ and (A.50) that

$$E\left[\left(\int_0^T |\hat{c}_t^{3,\varepsilon}|\,\nu(\mathrm{d}t)\right)^2\right] \le 2E\left[\left(\int_0^T |\hat{c}_t^3|\,\nu(\mathrm{d}t)\right)^2\right] + 2E\left[\left(\int_0^T |\Delta c_t^{3,\varepsilon}|\,\nu(\mathrm{d}t)\right)^2\right] < \infty.$$

Next, feasibility of $\hat{c}^3$ (with equality in the budget constraint), the definition of $\Delta c^{i,D,\varepsilon}$ and Lemma E.8 give

$$
\begin{aligned}
\int_0^T \frac{\hat{c}_t^{3,\varepsilon}}{B_t}\nu(\mathrm{d}t) &= \int_0^T \frac{\hat{c}_t^3}{B_t}\nu(\mathrm{d}t) - \int_0^T \frac{\Delta c_t^{1,D,\varepsilon}}{B_t}\nu(\mathrm{d}t) - \int_0^T \frac{\Delta c_t^{2,D,\varepsilon}}{B_t}\nu(\mathrm{d}t) \\
&= Y_T^3 + \varepsilon^i \bullet \|\hat{\varphi}^{1,\varepsilon}\|_T + \varepsilon^i \bullet \|\hat{\varphi}^{2,\varepsilon}\|_T = Y^3 + \Delta Y^\varepsilon.
\end{aligned}
$$

Next, using a similar estimate as in (A.63), it follows that

$$
\begin{aligned}
&E\left[\int_{[0,T]} -U_t^{3\prime\prime}(\hat{c}_t - |\Delta c_t^{3,\varepsilon}|)(\Delta c_t^{3,\varepsilon})^2\,\mu(\mathrm{d}t)\right] \\
&\quad = \hat{Z}_0^3 E^{\hat{Q}^3}\left[\int_{[0,T]} \gamma^3 \exp(\gamma^3|\Delta c_t^{3,\varepsilon}|)(\Delta c_t^{3,\varepsilon})^2\,\mu(\mathrm{d}t)\right] = o(\varepsilon^{\frac{2}{3}}).
\end{aligned}
$$

By [34, Remark 4.2], it remains to show that

$$E\left[\int_0^T U_t^{i\prime}(\hat{c}_t^3)(\hat{c}_t^{3,\varepsilon} - \hat{c}_t^3)\,\nu(\mathrm{d}t)\right] = \frac{1}{2}\varepsilon^{\frac{2}{3}}\hat{Z}_0^3 E^{\hat{Q}^3}\left[\int_0^T \mathfrak{E}_s\left(\frac{\alpha^1}{\Delta\overline{\varphi}_s^1}\Sigma_s^{\hat{\varphi}^1} + \frac{\alpha^2}{\Delta\overline{\varphi}_s^2}\Sigma_s^{\hat{\varphi}^2}\right)\mathrm{d}s\right] + o(\varepsilon^{\frac{2}{3}}).$$

This follows as in (A.60) using Bayes' theorem [34, Corollary A.2.(a)], Lemmas E.8, A.9, E.2 and A.6, as well as $\lim_{\varepsilon\to 0}\hat{Q}^3[\hat{\tau}_\varepsilon^i < T] = 0$ for $i \in \{1,2\}$ (this uses that $\hat{Q}^3 \approx \hat{Q}^1, \hat{Q}^2$).

B Proof of Stability Result

With the individual asymptotically optimal policies of Agents 1,2, and 3 at hand, we can now prove our main result, Theorem 4.1.

To this end, first notice that individual asymptotic optimality for all agents follows from Theorems 3.2 and 3.4. Moreover, the definitions of α^1 and α^2 yield $\alpha^1 + \alpha^2 = 1$. So it remains to verify that both the stock and goods market clear.

To establish stock market clearing, the crucial observation is that frictionless stock market clearing $\hat{\varphi}^1 + \hat{\varphi}^2$ implies that

$$\langle \hat{\varphi}^2 \rangle = \langle 1 - \hat{\varphi}^1 \rangle = \langle \hat{\varphi}^1 \rangle,$$

so that $\Sigma^{\hat{\varphi}^1} = \Sigma^{\hat{\varphi}^2}$. This has far reaching consequences. First, together with the definitions of α^1 and α^2 as well as $\Delta\overline{\varphi}^1$ and $\Delta\overline{\varphi}^2$ we obtain

$$\begin{aligned}\Delta\overline{\varphi}^1 &= \left(\frac{3}{2} \alpha^1 \mathfrak{E}_t \frac{A_t}{\gamma^1} \frac{\Sigma^{\hat{\varphi}^1}}{\Sigma^S} \right)^{\frac{1}{3}} = \left(\frac{3}{2} \mathfrak{E}_t \frac{A_t}{\gamma^1 + \gamma^2} \frac{\Sigma^{\hat{\varphi}^1}}{\Sigma^S} \right)^{\frac{1}{3}} \\ &= \left(\frac{3}{2} \mathfrak{E}_t \frac{A_t}{\gamma^1 + \gamma^2} \frac{\Sigma^{\hat{\varphi}^2}}{\Sigma^S} \right)^{\frac{1}{3}} = \left(\frac{3}{2} \alpha^2 \mathfrak{E}_t \frac{A_t}{\gamma^2} \frac{\Sigma^{\hat{\varphi}^2}}{\Sigma^S} \right)^{\frac{1}{3}} = \Delta\overline{\varphi}^2.\end{aligned}$$

Whence the (normalized) half-widths of the no-trade regions of agent 1 and 2 coincide. This means that one agent is willing to buy when the other one wants to sell and vice versa, so that

$$\Delta\varphi^{1,\varepsilon} = -\Delta\varphi^{2,\varepsilon}. \tag{B.1}$$

Together with the frictionless stock market clearing condition $\hat{\varphi}^1 + \hat{\varphi}^2 = 1$, this yields

$$\varphi^{1,\varepsilon} + \varphi^{2,\varepsilon} = 1.$$

This immediately gives (frictional) stock market clearing once we have shown that the respective liquidation times coincide:

$$\hat{\tau}^1_\varepsilon = \hat{\tau}^2_\varepsilon. \tag{B.2}$$

By the definition of $\hat{\tau}^1_\varepsilon$ and $\hat{\tau}^2_\varepsilon$ in (A.23), it suffices to argue that[32]

$$\tau^{1,D}_\varepsilon = \tau^{2,D}_\varepsilon. \tag{B.3}$$

and

$$\tau^{1,I}_\varepsilon = \tau^{2,I}_\varepsilon, \tag{B.4}$$

where the stopping times $\hat{\tau}^{1,D}_\varepsilon$ and $\hat{\tau}^{2,D}_\varepsilon$ are defined in (A.18), and $\hat{\tau}^{1,D}_\varepsilon$ and $\hat{\tau}^{2,D}_\varepsilon$ are defined in (A.21), respectively. Now (B.3) follows from $\|\varphi^{1,\varepsilon}\| = \|\varphi^{2,\varepsilon}\|$ by (B.2) and $|\hat{\varphi}^1 - \frac{1}{2}| = |1 - \hat{\varphi}^2 - \frac{1}{2}| = |-\hat{\varphi}^2 + \frac{1}{2}| = |\hat{\varphi}^2 - \frac{1}{2}|$, where we have used frictionless stock market clearing. Likewise (B.4) follows from $|\Delta\varphi^{1,\varepsilon} \bullet S| = |\Delta\varphi^{2,\varepsilon} \bullet S|$, where we have used (B.1).

To establish goods market clearing, use the frictionless goods market clearing $\hat{c}^1 + \hat{c}^2 + \hat{c}^3 =$

[32] The stopping time $\tau^{\mathfrak{E}}_\varepsilon$ defined in (A.5) is evidently the same for both agents.

$y^1+y^2+y^3+\delta$, the frictionless stock market clearing $\hat{\varphi}^1+\hat{\varphi}^2=1$ and the frictional stock market clearing $\hat{\varphi}^{1,\varepsilon}+\hat{\varphi}^{2,\varepsilon}=1$, obtaining

$$\begin{aligned}\hat{c}^{1,\varepsilon}+\hat{c}^{2,\varepsilon}+\hat{c}^{3,\varepsilon} &= \hat{c}^1-\frac{\varepsilon^1}{A}\bullet\|\hat{\varphi}^{1,\varepsilon}\|+\frac{\hat{\varphi}^{1,\varepsilon}-\hat{\varphi}^1}{A}\bullet S+\hat{c}^2-\frac{\varepsilon^2}{A}\bullet\|\hat{\varphi}^{2,\varepsilon}\|+\frac{\hat{\varphi}^{2,\varepsilon}-\hat{\varphi}^2}{A}\bullet S\\ &\quad+\hat{c}^3+\frac{\varepsilon^1}{A}\bullet\|\hat{\varphi}^{1,\varepsilon}\|+\frac{\varepsilon^2}{A}\bullet\|\hat{\varphi}^{2,\varepsilon}\|\\ &= \hat{c}^1+\hat{c}^2+\hat{c}^3+\frac{\hat{\varphi}^{1,\varepsilon}+\hat{\varphi}^{2,\varepsilon}-\hat{\varphi}^1-\hat{\varphi}^2}{A}\bullet S\\ &= \delta+\frac{1-1}{A}\bullet S=\delta.\end{aligned}$$

This completes the proof.

C Proof of Example

In this appendix, we show that the asset prices and strategies proposed in Section 5.2 form indeed an equilibrium for the economy from Section 5.1 . The starting point is the unique solution $r(t)$ of the integral equation (5.6), whose existence is established in Appendix D. It is used to define the martingales

$$\mathrm{d}M_t^i=-\lambda^S(t)\,\mathrm{d}W_t^\delta-\gamma^i\sqrt{\Sigma^{y^i}(t)}\,\mathrm{d}W_t^i+\gamma^i h(t)\sigma^{\xi^i}(t)\,\mathrm{d}W_t^\xi,\quad M_0^i=0,\quad i\in\{1,2\},\tag{C.1}$$

where the deterministic continuous function $h:[0,T]\to\mathbb{R}$ is given by

$$h(t)=\int_t^T\frac{A(u)}{A(t)}\lambda^S(u)\,\mathrm{d}u,\tag{C.2}$$

and the constants

$$S_0=A(0)\delta_0-\int_0^T A(t)(g^1(t)+g^2(t)+g^3(t))\,\mathrm{d}t,\tag{C.3}$$

$$\hat{c}_0^i=y_0^i+\hat{\varphi}_0^i\delta_0+\int_0^T\frac{A(t)}{A(0)}(g^i(t)-\hat{\varphi}_0^i(g^1(t)+g^2(t)+g^3(t))\,\mathrm{d}t,\quad i\in\{1,2\},\tag{C.4}$$

$$\hat{c}_0^3=y_0^3+\int_0^T\frac{A_t}{A_0}g_t^3\,\mathrm{d}t,\tag{C.5}$$

where the deterministic continuous functions $g^1,g^2,g^3:[0,T]\to\mathbb{R}$ are given by

$$\begin{aligned}g^i(t)&=\mu^{y^i}(t)-\lambda^S(t)\xi_0^i-h(t)\mu^{\xi^i}(t)+\frac{1}{2}\left(\frac{(\lambda^S(t))^2}{\gamma^i}-\gamma^i\Sigma^{y^i}(t)-\gamma^i h(t)^2\sigma^{\xi_i}(t)^2\right)\\ &\quad+\frac{\beta^i(t)-r(t)}{\gamma^i},\quad i\in\{1,2\},\end{aligned}\tag{C.6}$$

$$g^3(t)=\frac{\beta^3(t)-r(t)}{\gamma^3}.\tag{C.7}$$

C.1 Individual optimality

First, we check that the strategy/consumption pairs $(\hat{\varphi}^i, \hat{c}^i)$ given in (5.9), (5.10) and (C.4) are indeed individually optimal for Agents $i = 1, 2$, and the consumption rate $\hat{c}^3$ given in (5.11) and (C.5) is indeed individually optimal for Agent 3. We only spell out the argument for Agents $i = 1, 2$; the argument for Agent 3 is similar but easier. We use Lemma A.1 to verify the optimality of $(\hat{\varphi}^i, \hat{c}^i)$ for fixed $i \in \{1, 2\}$.[33]

Step 1. Define the martingale $\hat{Z}^i$ by

$$\hat{Z}^i_t = \gamma^i \exp(-\gamma^i \hat{c}^i_0)\mathcal{E}(M^i)_t \tag{C.8}$$

and the measure $\hat{Q}^i \approx P$ on $\mathcal{F}_T$ by

$$\mathrm{d}\hat{Q}^i = \frac{\hat{Z}^i_t}{\hat{Z}^i_0}\,\mathrm{d}P = \mathcal{E}(M^i)_T\,\mathrm{d}P.$$

By Girsanov's theorem and the fact that all integrands in the definition of M^i are continuous and bounded functions, it follows that $\hat{Q}^i \in \mathcal{M}^2(S)$.

Step 2. We next check that $\hat{\varphi}^i \bullet S$ is a $\hat{Q}^i$ martingale as required for Lemma A.1. By the definition of Σ^S and $\hat{\varphi}$ in (5.7) and (5.9), Bayes' theorem, Hölder's inequality (using also that Z^i is square integrable), the fact that A is decreasing, Jensen's inequality, and the fact that ξ^i is an inhomogeneous Brownian motion with drift, we obtain

$$\begin{aligned}
E^{\hat{Q}^i}\left[\langle \hat{\varphi}^i \bullet S\rangle_T\right]^2 &= E^{\hat{Q}^i}\left[\int_0^T (\hat{\varphi}^i_u)^2 \Sigma^S(u)\,\mathrm{d}u\right]^2 = E^{\hat{Q}^i}\left[\int_0^T A^2(u)\left(\frac{\lambda^S(u)}{\gamma^i} - \xi^i_u\right)^2\mathrm{d}u\right]^2 \\
&\leq A^2(0)E\left[(Z^i_T)^2\right]E\left[\left(\int_0^T \left(\frac{\lambda^S(u)}{\gamma^i} - \xi^i_u\right)^2 \mathrm{d}u\right)^2\right] \\
&\leq TA^2(0)E\left[(Z^i_T)^2\right]E\left[\int_0^T \left(\frac{\lambda^S(u)}{\gamma^i} - \xi^i_u\right)^4 \mathrm{d}u\right] < \infty.
\end{aligned}$$

The same calculation with $\hat{Z}^i$ replaced by Z, where $Z_T = \frac{\mathrm{d}Q}{\mathrm{d}P}$ for some $Q \in \mathcal{M}^2(S)$ shows that $\hat{\varphi}^i$ is admissible.

Step 3. We proceed to verify the first-order condition (A.1). The definition of U^i in (3.1), the dynamics of $\hat{c}^i$ in (5.10), and the definition of $\hat{Z}^i$ in (C.8) imply

$$\begin{aligned}
U^{i\prime}(\hat{c}^i_t) &= \gamma^i \exp\left(-\int_0^t \beta^i(u)\,\mathrm{d}u\right)\exp(-\gamma^i \hat{c}^i_t) \\
&= \gamma^i \exp(-\gamma^i \hat{c}^i_0)\exp\left(-\int_0^t \beta^i(u)\,\mathrm{d}u\right)\exp\left(-\gamma^i \int_0^t \mathrm{d}\hat{c}^i_u\right) \\
&= \gamma^i \exp(-\gamma^i \hat{c}^i_0)\exp\left(-\int_0^t \beta^i_u\,\mathrm{d}u\right)\exp\left(M_t - \frac{1}{2}\langle M\rangle_t + \int_0^t \beta^i(u)\,\mathrm{d}u - \int_0^t r(u)\,\mathrm{d}u\right) \\
&= \gamma^i \exp(-\gamma^i \hat{c}^i_0)\mathcal{E}(M)_t \frac{1}{B_t} = \frac{\hat{Z}^i_t}{B_t}.
\end{aligned}$$

[33] For Agent 3, one would instead use [34, Lemma 3.1].

Step 4. Finally, we check that the budget constraint (2.6) is satisfied with equality. Since $\frac{1}{B_t}\nu(\mathrm{d}t) = -\,\mathrm{d}A(t+)$, an integration by parts (using that $A(T+) = 0$ and $A(0+) = A(0)$) yields,

$$\begin{aligned}X_T^{\hat{\varphi}^i,\hat{c}^i,Y^i} &= \hat{\varphi}_0^i S_0 + \int_0^T \hat{\varphi}_t^i \,\mathrm{d}S_t + \int_0^T \frac{y_t^i - \hat{c}_t^i}{B_t}\,\nu(\mathrm{d}t)\\ &= \hat{\varphi}_0^i S_0 + \int_0^T \hat{\varphi}_t^i(\mu^S(t)\,\mathrm{d}t + \sqrt{\Sigma^S(t)}\,\mathrm{d}W_t) + A(0)(y_0^i - \hat{c}_0^i) + \int_0^T A(t)\,\mathrm{d}(y_t^i - \hat{c}_t^i). \end{aligned}\tag{C.9}$$

Plugging the definitions of $\hat{\varphi}^i$, Σ^S and μ^S in (5.9), (5.7) and (5.8) as well as the dynamics of y^i and $\hat{c}^i$ in (5.2) and (5.10) into (C.9) and collecting terms, we obtain

$$\begin{aligned}X_T^{\hat{\varphi}^i,\hat{c}^i,Y^i} &= \hat{\varphi}_0^i S_0 + A(0)(y_0^i - \hat{c}_0^i) + \int_0^T A(t)\left(\frac{\lambda^S(t)}{\gamma^i} - \xi_t^i\right)(\lambda^S(t)\,\mathrm{d}t + \,\mathrm{d}W_t^\delta)\\ &\quad + \int_0^T A(t)\Bigg(\mu^{y^i}(t)\,\mathrm{d}t + \sqrt{\Sigma^{y^i}(t)}\,\mathrm{d}W_t^i + \xi_t^i\,\mathrm{d}W_t^\delta\\ &\qquad\qquad + \frac{1}{\gamma^i}\left(\mathrm{d}M_t^i - \frac{1}{2}\langle M^i\rangle_t + (\beta^i(t) - r(t)\,\mathrm{d}t\right)\Bigg)\\ &= \hat{\varphi}_0^i S_0 + A(0)(y_0^i - \hat{c}_0^i) - \int_0^T A(t)\lambda^S(t)\xi_t^i\,\mathrm{d}t + \int_0^T A(t)h(t)\sigma^{\xi^i}(t)\,\mathrm{d}W_t^\xi\\ &\quad + \int_0^T A(t)\Big(\mu^{y^i}(t) + \frac{1}{2}\Big(\frac{\lambda^S(t)^2}{\gamma^i} - \gamma^i\Sigma^{y^i}(t) - \gamma^i(t)h^2(t)\sigma^{\xi_i}(t)^2\Big) + \frac{\beta^i(t) - r(t)}{\gamma^i}\Big)\,\mathrm{d}t.\end{aligned}$$

In turn, the identities $A(t)\lambda^S(t)\,\mathrm{d}t = -\,\mathrm{d}(A(t)h(t))$ and $\sigma^{\xi^i}(t)\,\mathrm{d}W^\xi = \mathrm{d}\xi_t^i - \mu_t^{\xi^i}\,\mathrm{d}t$ as well as an integration by parts yield

$$\begin{aligned}X_T^{\hat{\varphi}^i,\hat{c}^i,Y^i} &= \hat{\varphi}_0^i S_0 + A(0)(y_0^i - \hat{c}_0^i) + \int_0^T \xi_t^i\,\mathrm{d}(A(t)h(t)) + \int_0^T A(t)h(t)\,\mathrm{d}\xi_t^i - \int_0^T A(t)h(t)\mu_t^{\xi^i}\,\mathrm{d}t\\ &\quad + \int_0^T A(t)\left(\mu^{y^i}(t) + \frac{1}{2}\left(\frac{(\lambda^S(t))^2}{\gamma^i} - \gamma^i\Sigma^{y^i}(t) - \gamma^i h^2(t)\sigma^{\xi_i}(t)^2\right) + \frac{\beta^i(t) - r(t)}{\gamma^i}\right)\mathrm{d}t\\ &= \hat{\varphi}_0^i S_0 + A(0)(y_0^i - \hat{c}_0^i) - A(0)h(0)\xi_0^i - \int_0^T A(t)h(t)\mu^{\xi^i}(t)\,\mathrm{d}t\\ &\quad + \int_0^T A(t)\left(\mu^{y^i}(t) + \frac{1}{2}\Big(\frac{\lambda^S(t)^2}{\gamma^i} - \gamma^i\Sigma^{y^i}(t) - \gamma^i h^2(t)\sigma^{\xi_i}(t)^2\Big) + \frac{\beta^i(t) - r(t)}{\gamma^i}\right)\mathrm{d}t.\end{aligned}$$

Finally, using that $\hat{\varphi}_0^i S_0 + A(0)(y_0^i - \hat{c}_0^i) = \int_0^T A(t)g^i(t)\,\mathrm{d}t$ by the definition of S_0 and $\hat{c}_0^i$ in (C.3) and (C.4), noting that $A(0)h(0) = \int_0^T A(t)\lambda^S(t)\,\mathrm{d}t$, and recalling the definition of g^i in (C.6), we obtain

$$X_T^{\hat{\varphi}^i,\hat{c}^i,Y^i} = \int_0^T A(t)g^i(t)\,\mathrm{d}t - \int_0^T A(t)g^i(t)\,\mathrm{d}t = 0.$$

C.2 Stock and goods market clearing

We proceed to show that the stock and the goods market clear, which together with individual optimally shown above establishes that we have indeed an equilibrium.

Stock market clearing follows from the definition of $\hat{\varphi}^i$ in (5.9) via

$$\hat{\varphi}_t^1 + \hat{\varphi}_t^2 = \frac{1}{\sqrt{\Sigma^\delta(t)}} \left(\frac{\lambda^S(t)}{\gamma^1} + \frac{\lambda^S(t)}{\gamma^2} - \xi_t^1 - \xi_t^2 \right) = \frac{\sqrt{\Sigma^\delta(t)}}{\sqrt{\Sigma^\delta(t)}} = 1. \tag{C.10}$$

For the goods market, it suffices to show that

$$\hat{c}_0^1 + \hat{c}_0^2 + \hat{c}_0^3 = y_0^1 + y_0^2 + y_0^3 + \delta_0 \tag{C.11}$$

and

$$\mathrm{d}(\hat{c}_t^1 + \hat{c}_t^2 + \hat{c}_t^3) = \mathrm{d}(y_t^1 + y_t^2 + y_t^3 + \delta_t). \tag{C.12}$$

The equality (C.11) follows immediately from the definition of $\hat{c}_0^1, \hat{c}_0^2, \hat{c}_0^3$, and S_0 in (C.4), (C.5), and (C.3) together with initial stock market clearing ($\hat{\varphi}_0^1 + \hat{\varphi}_0^2 = 1$). To establish (C.12), we plug in the dynamics of $\hat{c}^1, \hat{c}^2, \hat{c}^3, \delta, y^1, y^2$ given in (5.10), (5.11), (5.1), (5.2) and use that y^3 is constant. We obtain

$$\begin{aligned}
&\mathrm{d}\big((\hat{c}_t^1 + \hat{c}_t^2 + \hat{c}_t^3 - (y_t^1 + y_t^2 + y_t^3 + \delta_t)\big) \\
&= \sum_{i=1}^2 \left(\frac{\lambda^S(t)}{\gamma^i} \,\mathrm{d}W_t^\delta + \sqrt{\Sigma^{y^i}(t)} \,\mathrm{d}W_t^i - h(t)\sigma^{\xi^i}(t) \,\mathrm{d}W_t^\xi \right) \\
&\quad - \sum_{i=1}^2 \left(\sqrt{\Sigma^{y^i}(t)} \,\mathrm{d}W_t^i - \xi_t^i \,\mathrm{d}W_t^\delta \right) - \sqrt{\Sigma^\delta(t)} \,\mathrm{d}W_t^\delta \\
&\quad + \sum_{i=1}^2 \frac{1}{2} \left(\frac{\lambda^S(t)^2}{\gamma^i} + \gamma^i \Sigma^{y^i}(t) + \gamma^i h^2(t) \sigma^{\xi^i}(t)^2 \right) \mathrm{d}t - \frac{\beta^i(t) - r(t)}{\gamma^i} \,\mathrm{d}t \\
&\quad - \frac{\beta^3(t) - r(t)}{\gamma^i} \,\mathrm{d}t - \sum_{i=1}^2 \mu^{y^i}(t) \,\mathrm{d}t - \mu^\delta(t) \,\mathrm{d}t.
\end{aligned}$$

Collecting terms and using that $\sigma^{\xi^1}(t) + \sigma^{\xi^2}(t) = 0$, we note that all martingale terms cancel. The remaining deterministic drift also vanishes because r satisfies the integral equation (5.6).

C.3 Applicability of the stability results

Finally, we check that the equilibrium quantities in Section 5.2 satisfy Assumptions A.2 and A.3 for constant transaction costs ($\mathfrak{C} = 1$). To this end, we verify the validity of Assumptions A.2 and A.3 for $\kappa = 2$. (Note that $Q^3 = P$.) For (A.2), the volatility-of-volatility of the spanned endowments needs to be small enough:

$$|\sigma^{\xi^i}(0)| + \int_0^T |\sigma^{\xi^i}{}'(u)| \,\mathrm{d}u < \left(\frac{1}{2} \frac{1}{2 \times 2 \times 4^2 \max(\gamma^1, \gamma^2)^2 T^2} \right)^{\frac{1}{2}}, \quad \text{for } i \in \{1, 2\}. \tag{C.13}$$

First, as $\mathfrak{C} = 1$, it follows that $\mu^{\mathfrak{C},Q} = 0$ under any $Q \approx P$ and $\Sigma^{\mathfrak{C}} = 0$.

Second, μ^S is by (5.8) continuous and deterministic and therefore the same for each $Q \approx P$, and $\Sigma^S = A^2 \Sigma^\delta$ by (5.7) is deterministic in C^1 and uniformly bounded above and away from zero.

Next, if follows from (5.9), (C.8), (C.1) and Girsanov's theorem that

$$\begin{aligned}
\mathrm{d}\hat{\varphi}^i &= -\frac{1}{2}\frac{\Sigma^{\delta\prime}(t)}{\Sigma^\delta(t)^{\frac{3}{2}}}\left(\frac{1}{\gamma^i}\lambda^S(t) - \xi^i_t\right)\mathrm{d}t + \frac{1}{\sqrt{\Sigma^\delta(t)}}\left(\frac{\lambda^{S\prime}(t)}{\gamma^i} - \mu^{\xi^i}(t)\right)\mathrm{d}t + \frac{\sigma^{\xi^i}(t)^2}{\sqrt{\Sigma^\delta(t)}}\gamma^i h(t)\,\mathrm{d}t \\
&\quad + \frac{\sigma^{\xi^i}(t)}{\sqrt{\Sigma^\delta(t)}}\,\mathrm{d}W^{\xi,\hat{Q}^i},
\end{aligned}$$

where W^{ξ,Q^i} is a $\hat{Q}^i$-Brownian motion. Note that the volatilities ξ^i of the spanned endowments are (inhomogeneous) $\hat{Q}^i$-Brownians motion with drift as well:

$$\mathrm{d}\xi^i = \big(\mu^{\xi^i}(t) + \sigma^{\xi^i}(t)^2\gamma^i h(t)\big)\,\mathrm{d}t + \sigma^{\xi^i}(t)\,\mathrm{d}W^{\xi,\hat{Q}^i}.$$

In particular, $\Sigma^{\hat{\varphi}^i}(t) = \sigma^{\xi^i}(t)^2/\sqrt{\Sigma^\delta(t)}$ is deterministic in C^1 and uniformly bounded above and away from zero, and $\mu^{\hat{\varphi}^i,P}$ and $\mu^{\hat{\varphi}^i,\hat{Q}^i}$ are the sum of a deterministic continuous function and a deterministic continuous function times an (inhomogeneous) Brownian motion (under P and $\hat{Q}^i$).

Finally, note that the normalised no-trade regions

$$\Delta\overline{\varphi}^i(t) = \left(\frac{3}{2}\frac{1}{(\gamma^1+\gamma^2)A(t)}\frac{\sigma^{\xi^i}(t)^2}{\Sigma^\delta(t)^2}\right)^{\frac{1}{3}}$$

are deterministic, in C^1, and uniformly bounded above and away from zero. In particular $\mu^{\Delta\overline{\varphi}^i}$ is continuous and deterministic and the same for each $Q \approx P$, and $\Sigma^{\Delta\overline{\varphi}^i} = 0$.

Putting all the above together, all conditions in A.2 and A.3 (for $\kappa = 2$) are immediately seen to be satisfied, apart from (A.2). To establish (A.2) (for $\kappa = 2$), first note that

$$\begin{aligned}
\frac{(\hat{\varphi}^i - \ell^i)^2}{A^2}\Sigma^S = (\hat{\varphi}^i - \ell^i)^2\Sigma^\delta &= \left(\frac{1}{\gamma^i}\lambda^S - \xi^i_0 - (\mu^{\xi^i} + (\sigma^{\xi^i})^2\gamma^i h)\bullet I - \sigma^{\xi^i}\bullet W^{\xi,\hat{Q}^i} - \sqrt{\Sigma^\delta}\ell^i\right)^2 \\
&= (f - \sigma^{\xi^i}\bullet W^{\xi,\hat{Q}^i})^2 \le 2f^2 + 2\left(\sigma^{\xi^i}\bullet W^{\xi,\hat{Q}^i}\right)^2,
\end{aligned}$$

where $f(t) := \frac{1}{\gamma^i}\lambda^S(t) - \xi^i_0 - \int_0^t \mu^{\xi^i}(u) + \sigma^{\xi^i}(u)^2\gamma^i h(u))\,\mathrm{d}u - \sqrt{\Sigma^\delta(t)}\ell^i$ is deterministic and continuous. Thus, it suffices to show that

$$E^{\hat{Q}^i}\left[\exp\left(2(\gamma^i)^2 4^2 \times 2\big(\sigma^{\xi^i}\bullet W^{\xi,\hat{Q}^i}\big)^2\bullet I_T\right)\right] < \infty. \tag{C.14}$$

To establish (C.14), apply an integration by parts to $\sigma^{\xi^i}\bullet W^{\xi,\hat{Q}^i}$ to obtain

$$\begin{aligned}
\left|\sigma^{\xi^i}\bullet W^{\xi,\hat{Q}^i}_t\right| &\le \left|W^{\xi,\hat{Q}^i}_t\sigma^{\xi^i}(0) + \int_0^t (W^{\xi,\hat{Q}^i}_t - W^{\xi,\hat{Q}^i}_u)\sigma^{\xi^i\prime}(u)\,\mathrm{d}u\right| \\
&\le \left|\sigma^{\xi^i}(0)\right|\left|W^{\xi,\hat{Q}^i}_t\right| + \int_0^t \left|\sigma^{\xi^i\prime}(u)\right|\mathrm{d}u \sup_{u\in[0,t]}|W^{\xi,\hat{Q}^i}_t - W^{\xi,\hat{Q}^i}_u| \\
&\le \left(\left|\sigma^{\xi^i}(0)\right| + \int_0^t \left|\sigma^{\xi^i\prime}(u)\right|\mathrm{d}u\right)\sup_{u\in[0,t]}|W^{\xi,\hat{Q}^i}_t - W^{\xi,\hat{Q}^i}_u| \qquad \text{(C.15)}
\end{aligned}$$

By the fact that $(W^{\xi,\hat{Q}^i}_t - W^{\xi,\hat{Q}^i}_u)_{u\in[0,t]}$ is again a $\hat{Q}^i$ Brownian motion and by the maximum principle for Brownian motion [38, Theorem 13.13], the right-hand side of (C.15) is equal in distribution to $\left(\left|\sigma^{\xi^i}(0)\right| + \int_0^t \left|\sigma^{\xi^i}{}'(u)\right| \mathrm{d}u\right) |W_t^{\xi,\hat{Q}^i}|$. Plugging this into (C.14) and using successively the maximum principle for Brownian motion, the fact that $W_T^{\xi,\hat{Q}^i}$ is equal in distribution to $\sqrt{T}\mathcal{N}$ (where $\mathcal{N}$ is standard normal distributed), and Assumption (C.13), we obtain

$$
\begin{aligned}
&E^{\hat{Q}^i}\left[\exp\left(2(\gamma^i)^2 4^2 \times 2\left(\sigma^{\xi^i} \bullet W^{\xi,\hat{Q}^i}\right)^2 \bullet I_T\right)\right] \\
&\quad\leq E^{\hat{Q}^i}\left[\exp\left(2(\gamma^i)^2 4^2 \times 2\left(\left|\sigma^{\xi^i}(0)\right| + \left|\sigma^{\xi^i}{}'(u)\right| \bullet I_T\right)^2 \sup_{t\in[0,T]}\left|W^{\xi,\hat{Q}^i}\right|^2 \bullet I_T\right)\right] \\
&\quad\leq E^{\hat{Q}^i}\left[\exp\left(2(\gamma^i)^2 4^2 \times 2\left(\left|\sigma^{\xi^i}(0)\right| + \left|\sigma^{\xi^i}{}'(u)\right| \bullet I_T\right)^2 \left|W_T^{\xi,\hat{Q}^i}\right|^2 T\right)\right] \\
&\quad\leq E\left[\exp\left(2(\gamma^i)^2 4^2 \times 2\left(\left|\sigma^{\xi^i}(0)\right| + \left|\sigma^{\xi^i}{}'(u)\right| \bullet I_T\right)^2 T^2\mathcal{N}^2\right)\right] \\
&\quad< \infty,
\end{aligned}
$$

where we have used in the last step that $E\left[\exp(\alpha\mathcal{N}^2)\right] < \infty$ for $\alpha < \frac{1}{2}$. This gives (C.14).

D An integral equation

Consider $f_1, f_2, f_3 \in C[0,T]$ and a finite measure ν on $[0,T]$ with $\nu[[0,t]) > 0$ for all $t \in (0,T]$. For a function $x \in L^1[0,T]$, define the operators $I, J, K : L^1[0,T] \to L^1[0,T]$ by

$$Ix(t) = \int_0^t x(u)\,\mathrm{d}u, \tag{D.1}$$

$$Jx(t) = \int_{[0,t)} \exp(Ix(u))\,\nu(\mathrm{d}u), \tag{D.2}$$

$$Kx(t) = f_1(t) + f_2(t)\left(\int_0^t \frac{Jx(u)}{Jx(t)} f_3(u)\,\mathrm{d}u\right)^2. \tag{D.3}$$

Note that for each $x \in L^1[0,T]$, Jx is nondecreasing and null at 0. This implies that Kx is well defined with $Kx(0) = f_1(0)$ and satisfies the a priori estimate

$$|Kx(t)| \leq |f_1(t)| + |f_2(t)|\left(\int_0^t |f_3(u)|\,\mathrm{d}u\right)^2. \tag{D.4}$$

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$$x(t) = Kx(t), \quad t \in [0,T], \tag{D.5}$$

has a unique solution, which is even in $C[0,T]$.

Proof. Set $B_K := \int_0^T \left(|f_1(t)| + |f_2(t)|\left(\int_0^t |f_3(u)|\,\mathrm{d}u\right)^2\right)\mathrm{d}t < \infty$ and

$$\mathcal{B}_K := \left\{x \in L^1[0,T] : \int_0^T |x(u)|\,\mathrm{d}u \leq B_K\right\}.$$

It follows from the priori estimate (D.4) that K maps $\mathcal{B}_K$ into itself. This implies that if the integral equation (D.5) has a solution x, then necessarily $x \in \mathcal{B}_K$. Moreover, x is then automatically continuous as the right hand side of (D.5) is continuous.

We proceed to show that (D.5) has a unique solution in $\mathcal{B}_K$.

For $a \in \mathbb{R}_+$, define on $L^1[0,T]$ the equivalent norm $\|\cdot\|_a$ by

$$\|x\|_a = \int_0^T |x(u)| \exp(-au)\,\mathrm{d}u.$$

Then $\mathcal{B}_K$ endowed with the norm $\|\cdot\|_a$ is a Banach space. By Banach's fixed-point theorem, it suffices to show that for a suitable choice of a, the operator K under the norm $\|\cdot\|_a$ is a contraction on $\mathcal{B}_K$.

Let $x_1, x_2 \in \mathcal{B}_K$, set $\Delta x := |x_1 - x_2|$, and note that

$$I\Delta x(T) = \int_0^T |x_1(u) - x_2(u)|\,\mathrm{d}u \leq 2B_K$$

Moreover,

$$Ix_1(t) - I\Delta x(t) \leq Ix_2(t) \leq Ix_1(t) + I\Delta x(t), \quad t \in [0,T].$$

Since $t \mapsto I\Delta x(t)$ is nondecreasing, this estimate gives

$$Jx_1(t)\exp(-I\Delta x(t)) \leq Jx_2(t) \leq Jx_1(t)\exp(I\Delta x(t)), \quad t \in [0,T],$$

and hence,

$$\frac{Jx_1(t)}{Jx_2(t)} \leq \exp(I\Delta x(t)) \quad \text{and} \quad \frac{Jx_2(t)}{Jx_1(t)} \leq \exp(I\Delta x(t)), \quad t \in (0,T].$$

Thus, for $0 < u \leq t$, using that $t \mapsto I\Delta x(t)$ and $t \mapsto Jx(t)$ are nondecreasing and that for fixed $w > 0$, $\exp(v) - 1 \leq \exp(w)v$ for all $v \in [0,w]$, we obtain

$$\begin{aligned}
\left|\frac{Jx_1(u)}{Jx_1(t)} - \frac{Jx_2(u)}{Jx_2(t)}\right| &= \mathbf{1}_{\left\{\frac{Jx_1(u)}{Jx_1(t)} > \frac{Jx_2(u)}{Jx_2(t)}\right\}} \left(\frac{Jx_1(u)}{Jx_2(u)}\frac{Jx_2(t)}{Jx_1(t)} - 1\right)\frac{Jx_2(u)}{Jx_2(t)} \\
&\quad + \mathbf{1}_{\left\{\frac{Jx_2(u)}{Jx_2(t)} \geq \frac{Jx_1(u)}{Jx_1(t)}\right\}} \left(\frac{Jx_2(u)}{Jx_1(u)}\frac{Jx_1(t)}{Jx_2(t)} - 1\right)\frac{Jx_1(u)}{Jx_1(t)} \\
&\leq \mathbf{1}_{\left\{\frac{Jx_1(u)}{Jx_1(t)} > \frac{Jx_2(u)}{Jx_2(t)}\right\}} \big(\exp(I\Delta x(u) + I\Delta x(t)) - 1\big)\frac{Jx_2(u)}{Jx_2(t)} \\
&\quad + \mathbf{1}_{\left\{\frac{Jx_2(u)}{Jx_2(t)} \geq \frac{Jx_1(u)}{Jx_1(t)}\right\}} \big(\exp(I\Delta x(u) + I\Delta x(t)) - 1\big)\frac{Jx_1(u)}{Jx_1(t)} \\
&\leq \mathbf{1}_{\left\{\frac{Jx_1(u)}{Jx_1(t)} > \frac{Jx_2(u)}{Jx_2(t)}\right\}} \big(\exp(2I\Delta x(t)) - 1\big) \\
&\quad + \mathbf{1}_{\left\{\frac{Jx_2(u)}{Jx_2(t)} \geq \frac{Jx_1(u)}{Jx_1(t)}\right\}} \big(\exp(2I\Delta x(t)) - 1\big) \\
&\leq \exp(2I\Delta x(t)) - 1 \\
&\leq \exp(2I\Delta x(T)) \times 2I\Delta x(t) \\
&\leq 2\exp(4B_K) I\Delta x(t).
\end{aligned} \tag{D.6}$$

As $t \mapsto Jx(t)$ is nondecreasing, this yields

$$\begin{aligned}
|Kx_1(t) - Kx_2(t)| &= |f_2(t)| \left| \int_0^t \left(\frac{Jx_1(u)}{Jx_1(t)} + \frac{Jx_2(u)}{Jx_2(t)} \right) f_3(u) \, \mathrm{d}u \right| \\
&\quad \times \left| \int_0^t \left(\frac{Jx_1(u)}{Jx_1(t)} - \frac{Jx_2(u)}{J_2x(t)} \right) f_3(u) \, \mathrm{d}u \right| \\
&\leq |f_2(t)| \left(\int_0^t 2|f_3(u)| \, \mathrm{d}u \right) \left(2 \exp(4B_K) I\Delta x(t) \int_0^t |f_3(u)| \, \mathrm{d}u \right) \\
&\leq 4 \exp(4B_K) I\Delta x(t) |f_2(t)| t^2 \sup_{0 \leq u \leq t} |f_3(u)|^2 \\
&\leq 4T^2 \exp(4B_K) \sup_{0 \leq u \leq T} |f_2(u)||f_3(u)|^2 I\Delta x(t) \\
&=: C_K I\Delta x(t).
\end{aligned} \tag{D.7}$$

Now for $a := 2C_K$, by the above and Fubini's theorem,

$$\begin{aligned}
\|Kx_1 - Kx_2\|_a &= \int_0^T |Kx_1(u) - Kx_2(u)| \exp(-au) \, \mathrm{d}u \\
&\leq C_K \int_0^T I\Delta x(u) \exp(-au) \, \mathrm{d}u \\
&= C_K \int_0^T \int_0^u \Delta x(s) \, \mathrm{d}s \exp(-au) \, \mathrm{d}u \\
&= C_K \int_0^T \Delta x(s) \int_s^T \exp(-au) \, \mathrm{d}u \, \mathrm{d}s \\
&= C_K \int_0^T \Delta x(s) \frac{\exp(-as) - \exp(-at)}{a} \, \mathrm{d}s \\
&\leq \frac{C_k}{a} \int_0^T \Delta x(s) \exp(-as) \, \mathrm{d}s \\
&= \frac{1}{2} \|\Delta x\|_a = \frac{1}{2} \|x_1 - x_2\|_a.
\end{aligned}$$

This shows that the mapping $x \mapsto Kx$ is indeed a contraction on $\mathcal{B}_K$ with respect to the norm $\| \cdot \|_a$, thereby completing the proof. □

E Facts from stochastic analysis

In this appendix, we collect several facts and auxiliary results from stochastic analysis.

E.1 $\mathcal{S}^p$ and $\mathcal{H}^p$ norms.

First, we recall the fundamental $\mathcal{S}^p$- and $\mathcal{H}^p$-norms for semimartingales (cf. [57] for more details):

Definition E.1. Let $p \in [1, \infty]$ and $Q \approx P$ on $\mathcal{F}_T$. For an adapted càdlàg process $(X_t)_{t \in [0,T]}$, the $\mathcal{S}^p$-norm of X with respect to Q is defined as

$$\|X\|_{\mathcal{S}^p(Q)} := \|X_t^*\|_{L^p(Q)}.$$

Stochastic processes converging in $\mathcal{S}^1(Q)$ converge a fortiori at the time horizon in $L^1(Q)$. The following simple result shows that $\mathcal{S}^1(Q)$ convergence even implies that the processes converge in $L^1(Q)$ at stopping times converging "stationarily" to the time horizon in probability.

Lemma E.2. *Let $Q \approx P$ on $\mathcal{F}_T$ be an equivalent measure and $\big((X_t^\varepsilon)_{t\in[0,T]}\big)_{\varepsilon>0}$ a family of càdlàg processes converging in $\mathcal{S}^1(Q)$ to the adapted càdlàg process $(X_t)_{t\in[0,T]}$ as $\varepsilon \downarrow 0$. Moreover, let $(\tau_\varepsilon)_{\varepsilon>0}$ be a family of stopping times satisfying $\lim_{\varepsilon\downarrow 0} Q[\tau_\varepsilon < T] = 0$. Then:*

$$\lim_{\varepsilon\downarrow 0} E^Q\left[X^\varepsilon_{\tau_\varepsilon}\right] = E^Q\left[X_T\right]. \tag{E.1}$$

Proof. Convergence of X^ε to X in $\mathcal{S}^1(Q)$ implies in particular that X_T^ε converges in $L^1(Q)$ to X_T as $\varepsilon \downarrow 0$. Thus, it suffices to show that $X_T^\varepsilon - X^\varepsilon_{\tau_\varepsilon}$ converges in $L^1(Q)$ to 0. As $(X^\varepsilon)^*_T$ converges in $L^1(Q)$, there is $\varepsilon_0 > 0$, such that for each countable subset I of $(0, \varepsilon_0]$, the countable subfamily $\big((X^\varepsilon)^*_T\big)_{\varepsilon\in I}$ is uniformly integrable under Q. As $X_T^\varepsilon - X^\varepsilon_{\tau_\varepsilon}$ converges to 0 in probability and $|X_T^\varepsilon - X^\varepsilon_{\tau_\varepsilon}| \leq 2(X^\varepsilon)^*_T$, the result follows from the dominated convergence theorem. $\square$

Definition E.3. Let $p \in [1,\infty)$ and $Q \approx P$ on $\mathcal{F}_T$. For an Itô process $(X_t)_{t\in[0,T]}$ null at 0 with drift rate $\mu^{X,Q}$ (under Q) and instantaneous variance Σ^X, the $\mathcal{H}^p$-norm of X with respect to Q is defined as[34]

$$\|X\|_{\mathcal{H}^p(Q)} := \left\| \left|\mu^{X,Q}\right| \bullet I_T + (\Sigma^X \bullet I_T)^{1/2} \right\|_{L^p(Q)}.$$

The following fundamental inequality [57, Theorems V.2 and Theorem V.3] connects the S^p and H^p norm. It is used repeatedly in various estimates in Appendix A.

Lemma E.4. *Let $p \in [0,\infty)$ and consider an Itô process $(X_t)_{t\in[0,T]}$ and $H \in L(X)$. Then:*

$$\|H \bullet X\|_{\mathcal{S}^p} \leq C_p \|H\|_{\mathcal{S}^\infty} \|X\|_{\mathcal{H}^p}, \tag{E.2}$$

where C_p is a universal constant only depending on p.

E.2 Results from stochastic analysis

The following criterion for local supermartingales to be (true) supermartingales is folklore, but for lack of references we include its proof:

Lemma E.5. *Let $(M_t)_{t\in[0,T]}$ be a continuous local martingale and $(A_t)_{t\in[0,T]}$ a nondecreasing process null at zero. Then $X := M - A$ is a supermartingale if and only if M is a supermartingale and $E\left[A_T\right] < \infty$.*

Proof. If M is a supermartingale and $E\left[A_T\right] < \infty$, then X is integrable and the supermartingale property follows from the supermartingale property of M and the fact that A is nondecreasing. Conversely, if X is a supermartingale, then $X^- = \max(-X, 0)$ is a submartingale. Hence it is of class D by [20, Theorem VI.22b)]. Since $M^- \leq X^-$, we may deduce that M^- is of class D, too.

[34] To be precise, this is a slight abuse of notation, as the usual textbook definition of the $\mathcal{H}^p(Q)$-norm is the minimum of $\left\| \|A\|_T + [M]_T^{1/2} \right\|_{L^p(\hat{Q})}$ over all decompositions $X = M + A$ into a local Q-martingale M and a càdlàg adapted process of finite variation A. It is not clear that the minimum is attained for the canonical decomposition of X (under Q). However, this is not a problem as both norms are equivalent; see [51].

Now it follows from [25, Proposition 2.2] that M is a supermartingale. As supermartingales are integrable, $A_T = M_T - X_T$ is integrable as well. $\square$

Doob's inequality gives rise to the following "exponential BDG inequality" which is used in the proof of Theorem 3.2.

Lemma E.6. *Let $(M_t)_{t\in[0,T]}$ be a continuous local martingale. Then*

$$E\left[\exp(M_T^*)\right] \leq 8E\left[\exp\left(2\langle M\rangle_T\right)\right]$$

Proof. Applying Doob's inequality [20, Theorem V.24] to the (local) submartingale $E\left[\exp(|M|)\right]$ gives

$$E\left[\exp\left(M_t^*\right)\right] \leq 4E\left[\exp(|M_T|)\right].$$

Now the result follows from the elementary estimate $\exp(|M_T|) \leq \exp(M_t) + \exp(-M_t)$ and [57, Theorem III.43]. $\square$

The following simple consequence of Markov's inequality is used repeatedly in the estimates required for the proof of Theorem 3.2.

Lemma E.7. *Fix $p \geq 1$, $q > r \in \mathbb{R}$, $\alpha > 0$ and let $Q \approx P$ be an equivalent measure. For $\varepsilon > 0$, let $(K_t^\varepsilon)_{t\in[0,T]}$ be an adapted càdlàg process. For $\varepsilon > 0$, define the stopping time*

$$\tau_\varepsilon = \inf\left\{|\varepsilon^q K_t^\varepsilon| > \alpha\varepsilon^r\right\} \wedge T.$$

Suppose that one of the following conditions is satisfied:

(a) *There is a nonnegative adapted càdlàg process $(L_t)_{t\in[0,T]}$ with $L_T^* \in L^p(Q)$ and $|K^\varepsilon| \leq L$ for all $\varepsilon > 0$ sufficiently small.*

(b) *There is a nonnegative adapted càdlàg process $(L_t)_{t\in[0,T]}$ with $L_T^* \in L^p(Q)$ and K^ε converges to L in $\mathcal{S}^p$.*

(c) *Each K^ε is a local martingale, and there is a a nonnegative and nondecreasing adapted càdlàg process $(L_t)_{t\in[0,T]}$ with $L_T \in L^{p/2}(Q)$ and $[K^\varepsilon] \leq L$ for all $\varepsilon > 0$ sufficiently small.*

Then:

$$Q[\tau_\varepsilon < T] = O(\varepsilon^{p(q-r)}).$$

Proof. By the definition of τ_ε and Markov's inequality,

$$\begin{aligned} Q[\tau_\varepsilon < T] &\leq Q[\varepsilon^q (K^\varepsilon)_T^* > \alpha\varepsilon^r] = Q[(K^\varepsilon)_T^* > \alpha\varepsilon^{r-q}] = Q[\left((K^\varepsilon)_T^*\right)^p > \alpha^p\varepsilon^{p(r-q)}] \\ &\leq \varepsilon^{p(q-r)}\alpha^{-p}E^Q\left[\left((K^\varepsilon)_T^*\right)^p\right]. \end{aligned}$$

Given (a), the claim follows from $\limsup_{\varepsilon\downarrow 0} E^Q\left[\left((K^\varepsilon)_T^*\right)^p\right] \leq E^Q\left[(L_T^*)^p\right]$. Given (b), the assertion is a consequence of $\lim_{\varepsilon\downarrow 0} E^Q\left[\left((K^\varepsilon)_T^*\right)^p\right] = E^Q\left[(L_T^*)^p\right]$. Given (c), the claim is implied by

$$\limsup_{\varepsilon\downarrow 0} E^Q\left[\left((K^\varepsilon)_T^*\right)^p\right] \leq \limsup_{\varepsilon\downarrow 0} C_p E^Q\left[\left([K^\varepsilon]_T\right)^{p/2}\right] \leq C_p E^Q\left[L_T^{p/2}\right],$$

where C_p is the constant in the Burkholder-Davis-Gundy inequality [38, Theorem 26.12]. $\square$

E.3 A result on the annuity process

The following auxiliary result shows how to use the annuity process A to construct an adapted consumption stream that matches a given (discounted) cumulative endowment at maturity T:

Lemma E.8. *Let $(\Upsilon_t)_{t\in[0,T]}$ be an adapted làdlàg process. Define the process $(c^\Upsilon)_{t\in[0,T]}$ by*

$$c_t^\Upsilon := \frac{\Upsilon_0}{A_0} + \int_0^t \frac{1}{A_t}\,\mathrm{d}\Upsilon_t.$$

Then:

$$(c^\Upsilon)_T^* \leq \frac{2}{A_T}\Upsilon_T^*, \tag{E.3}$$

and

$$\int_0^T \frac{c_t^\Upsilon}{B_t}\,\nu(\mathrm{d}t) = \Upsilon_T. \tag{E.4}$$

Proof. First, an integration by parts gives

$$c_t^\Upsilon = \frac{\Upsilon_t}{A_t} - \int_0^t \Upsilon_t\,\mathrm{d}\left(\frac{1}{A}\right)_t, \quad t \in [0,T].$$

Taking absolute values and using that $\frac{1}{A}$ is nonnegative and nondecreasing gives (E.3) via

$$\left|\frac{\Upsilon_t}{A_t} - \int_0^t \Upsilon_t\,\mathrm{d}\left(\frac{1}{A}\right)_t\right| \leq \frac{\Upsilon_t^*}{A_t} + \Upsilon_t^*\left(\frac{1}{A_t} - \frac{1}{A_0}\right) \leq \frac{2}{A_T}\Upsilon_T^*, \qquad t \in [0,T].$$

Next, using the definition of the annuity process A and integrating by parts (noting that $A_{T+} =: 0$ and $A_{0+} = A_0$), we obtain (E.4) via.

$$\begin{aligned}\int_0^T \frac{c_t^\Upsilon}{B_t}\,\nu(\mathrm{d}t) &= -\int_0^T c_t^\Upsilon\,\mathrm{d}A_{t+} = -c_T^\Upsilon A_{T+} + c_0^\Upsilon A_0 + \int_{(0,T]} A_t\,\mathrm{d}c_t^\Upsilon \\ &= \Upsilon_0 + \int_{(0,T]} \mathrm{d}\Upsilon_t = \Upsilon_0 + (\Upsilon_T - \Upsilon_0) = \Upsilon_T.\end{aligned}$$

□

References

[1] L. Ahrens. *On Using Shadow Prices for the Asymptotic Analysis of Portfolio Optimization under Proportional Transaction Costs.* PhD thesis, Christian-Albrechts-Universität zu Kiel, 2015.

[2] R. F. Almgren and N. Chriss. Optimal execution of portfolio transactions. *J. Risk*, 3:5–40, 2001.

[3] R. F. Almgren and T. M. Li. Option hedging with smooth market impact. *Market Microstucture and Liquidity*, to appear, 2016.

[4] A. Altarovici, J. Muhle-Karbe, and H. M. Soner. Asymptotics for fixed transaction costs. *Finance Stoch.*, 19(2):363–414, 2015.

[5] S. Biagini and M. Frittelli. Utility maximization in incomplete markets for unbounded processes. *Finance Stoch.*, 9(4):493–517, 2005.

[6] S. Biagini and M. Frittelli. The supermartingale property of the optimal wealth process for general semimartingales. *Finance Stoch.*, 11(2):253–266, 2007.

[7] B. Bouchard, M. Fukasawa, M. Herdegen, and J. Muhle-Karbe. Equilibrium returns with transaction costs. Preprint, 2017.

[8] A. Buss and B. Dumas. Trading fees and slow-moving capital. Preprint, 2015.

[9] A. Buss, R. Uppal, and G. Vilkov. Asset prices in general equilibrium with recursive utility and illiquidity induced by transactions costs. Preprint, 2013.

[10] J. Cai, M. Rosenbaum, and P. Tankov. Asymptotic lower bounds for optimal tracking: a linear programming approach. *Ann. Appl. Probab.*, 2015. to appear.

[11] J. Cai, M. Rosenbaum, and P. Tankov. Asymptotic optimal tracking: feedback strategies. *Stochastics*, 2016. to appear.

[12] J. H. Choi and K. Larsen. Taylor approximation of incomplete Radner equilibrium models. *Finance Stoch.*, 19(3):653–679, 2015.

[13] P. O. Christensen and K. Larsen. Incomplete continuous-time securities markets with stochastic income volatility. *Rev. Asset Pric. Stud.*, 4(2):247–285, 2014.

[14] P. O. Christensen, K. Larsen, and C. Munk. Equilibrium in securities markets with heterogeneous investors and unspanned income risk. *J. Econ. Theory*, 147(3):1035–1063, 2012.

[15] G. M. Constantinides. Capital market equilibrium with transaction costs. *J. Pol. Econ.*, 94(4):842–862, 1986.

[16] J. Cvitanić and I. Karatzas. Hedging and portfolio optimization under transaction costs: a martingale approach. *Math. Finance*, 6(2):133–165, 1996.

[17] E. Davila. Optimal financial transaction taxes. Preprint, 2014.

[18] M. H. A. Davis and A. R. Norman. Portfolio selection with transaction costs. *Math. Oper. Res.*, 15(4):676–713, 1990.

[19] F. Delbaen, P. Grandits, T. Rheinländer, D. Samperi, M. Schweizer, and C. Stricker. Exponential hedging and entropic penalties. *Math. Finance*, 12(2):99–123, 2002.

[20] C. Dellacherie and P.-A. Meyer. *Probabilities and Potential B: Theory of Martingales*. North-Holland, Amsterdam, 1982.

[21] A. Dixit. Analytical approximations in models of hysteresis. *Rev. Econ. Stud.*, 58(1):141–151, 1991.

[22] D. Duffie. *Dynamic asset pricing theory*. Princeton University Press,Princeton, N.J., third edition, 2001.

[23] D. Duffie and W. Zame. The consumption-based capital asset pricing model. *Econometrica*, 57(6):1279–1297, 1989.

[24] B. Dumas and E. Luciano. An exact solution to a dynamic portfolio choice problem under transactions costs. *J. Finance*, 46(2):577–595, 1991.

[25] K. D. Elworthy, X.-M. Li, and M. Yor. The importance of strictly local martingales; applications to radial Ornstein-Uhlenbeck processes. *Probab. Theory Related Fields*, 115(3):325–355, 1999.

[26] M. R. Feodoria. *Optimal investment and utility indifference pricing in the presence of small fixed transaction costs*. PhD thesis, Christian-Albrechts-Universität zu Kiel, 2016.

[27] M. Frittelli. The minimal entropy martingale measure and the valuation problem in incomplete markets. *Math. Finance*, 10(1):39–52, 2000.

[28] N. Garleanu and L. H. Pedersen. Dynamic trading with predictable returns and transaction costs. *J. Finance*, 68(6):2309–2340, 2013.

[29] N. Garleanu and L. H. Pedersen. Dynamic portfolio choice with frictions. *J. Econ. Theory*, 165:487–516, 2016.

[30] S. Gerhold, P. Guasoni, J. Muhle-Karbe, and W. Schachermayer. Transaction costs, trading volume, and the liquidity premium. *Fin. Stoch.*, 18(1):1–37, 2014.

[31] P. Guasoni and M. Weber. Nonlinear price impact and portfolio choice. Preprint, 2015.

[32] J. Heaton and D. Lucas. Evaluating the effects of incomplete markets on risk sharing and asset pricing. *J. Polit. Econ.*, 104(3):443–487, 1993.

[33] V. Henderson. Valuation of claims on nontraded assets using utility maximization. *Math. Finance*, 12(4):351–373, 2002.

[34] M. Herdegen and J. Muhle-Karbe. Sensitivity of optimal consumption streams. Preprint, 2015.

[35] K. Janeček and S. E. Shreve. Asymptotic analysis for optimal investment and consumption with transaction costs. *Finance Stoch.*, 8(2):181–206, 2004.

[36] K. Janeček and S. E. Shreve. Futures trading with transaction costs. *Illinois J. Math.*, 54(4):1239–1284, 2010.

[37] Y. Kabanov and C. Stricker. On the optimal portfolio for the exponential utility maximization: remarks to the six-author paper. *Math. Finance*, 12(2):125–134, 2002.

[38] O. Kallenberg. *Foundations of Modern Probability.* Springer, New York, second edition, 2002.

[39] J. Kallsen and S. Li. Portfolio optimization under small transaction costs: a convex duality approach. Preprint, 2013.

[40] J. Kallsen and J. Muhle-Karbe. On using shadow prices in portfolio optimization with transaction costs. *Ann. Appl. Probab.*, 20(4):1341–1358, 2010.

[41] J. Kallsen and J. Muhle-Karbe. Option pricing and hedging with small transaction costs. *Math. Finance*, 25(4):702–723, 2015.

[42] J. Kallsen and J. Muhle-Karbe. The general structure of optimal investment and consumption with small transaction costs. *Math. Finance*, 27(3):659–703, 2017.

[43] I. Karatzas, J. P. Lehoczky, and S. E. Shreve. Equilibrium models with singular asset prices. *Math. Finance*, 1(3):11–29, 1991.

[44] I. Karatzas and S. E. Shreve. *Methods of mathematical finance.* Springer, New York, 1998.

[45] C. Kardaras, H. Xing, and G. Žitković. Incomplete stochastic equilibria with exponential utilities: close to Pareto optimality. Preprint, 2015.

[46] R. Korn. Portfolio optimisation with strictly positive transaction costs and impulse control. *Finance Stoch.*, 2(2):85–114, 1998.

[47] H. Liu and M. Loewenstein. Optimal portfolio selection with transaction costs and finite horizons. *Rev. Financ. Stud.*, 15(3):805–835, 2002.

[48] A. Lo, H. Mamaysky, and J. Wang. Asset prices and trading volume under fixed transactions costs. *J. Polit. Econ.*, 112(5):1054–1090, 2004.

[49] M. J. P. Magill and G. M. Constantinides. Portfolio selection with transactions costs. *J. Econ. Theory*, 13(2):245–263, 1976.

[50] R. Martin. Optimal trading under proportional transaction costs. *RISK*, August:54–59, 2014.

[51] P.-A. Meyer. Inégalités de normes pour les intégrales stochastiques. *Sém. Probab. Strasbourg*, 12:757–762, 1978.

[52] L. Moreau, J. Muhle-Karbe, and H. M. Soner. Trading with small price impact. *Math. Finance*, 27(2):350–400, 2017.

[53] M. P. Owen. Utility based optimal hedging in incomplete markets. *Ann. Appl. Probab.*, 12(2):691–709, 2002.

[54] M. P. Owen and G. Žitković. Optimal investment with an unbounded random endowment and utility-based pricing. *Math. Finance*, 19(1):129–159, 2009.

[55] L. H. Pedersen. Frictional finance. Presentation, available at `http://docs.lhpedersen.com/FrictionalFinance.pdf`, 2010.

[56] D. Possamai, H. M. Soner, and N. Touzi. Homogenization and asymptotics for small transaction costs: the multidimensional case. *Comm. Part. Diff. Eq.*, 40(11):2005–2046, 2015.

[57] P. E. Protter. *Stochastic integration and differential equations.* Springer, Berlin, second edition, 2005.

[58] R. Radner. Existence of equilibrium of plans, prices, and price expectations in a sequence of markets. *Econometrica*, pages 289–303, 1972.

[59] R. Radner. Collusive behavior in noncooperative epsilon-equilibria of oligopolies with long but finite lives. *J. Econ. Theory*, 22(2):136–154, 1980.

[60] L. C. G. Rogers. Why is the effect of proportional transaction costs $O(\delta^{2/3})$? In *Mathematics of Finance*, pages 303–308. Amer. Math. Soc., Providence, RI, 2004.

[61] M. Rosenbaum and P. Tankov. Asymptotically optimal discretization of hedging strategies with jumps. *Ann. Appl. Probab.*, 24(3):1002–1048, 2014.

[62] W. Schachermayer. Optimal investment in incomplete markets when wealth may become negative. *Ann. Appl. Probab.*, 11(3):694–734, 2001.

[63] W. Schachermayer. A super-martingale property of the optimal portfolio process. *Finance Stoch.*, 7(4):433–456, 2003.

[64] S. E. Shreve and H. M. Soner. Optimal investment and consumption with transaction costs. *Ann. Appl. Probab.*, 4(3):609–692, 1994.

[65] H. M. Soner and N. Touzi. Homogenization and asymptotics for small transaction costs. *SIAM J. Control Optim.*, 51(4):2893–2921, 2013.

[66] D. Vayanos. Transaction costs and asset prices: a dynamic equilibrium model. *Rev. Financ. Stud.*, 11(1):1–58, 1998.

[67] D. Vayanos and J.-L. Vila. Equilibrium interest rate and liquidity premium with transaction costs. *Econ. Theory*, 13(3):509–539, 1999.

[68] J. Wang. A model of intertemporal asset prices under asymmetric information. *Rev. Econ. Stud.*, 60(2):249–282, 1993.

[69] K. Weston. Existence of a Radner equilibrium in a model with transaction costs. Preprint, 2017.

[70] A. E. Whalley and P. Wilmott. An asymptotic analysis of an optimal hedging model for option pricing with transaction costs. *Math. Finance*, 7(3):307–324, 1997.

[71] G. Žitković. An example of a stochastic equilibrium with incomplete markets. *Finance Stoch.*, 16(2):177–206, 2012.